- Quant Methods -

Fix the VIX: Reducing Manipulation in the Volatility Index

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Thursday, April 12, 2018

By Joe Pimbley and Gene Phillips

On February 5, 2018, the CBOE Volatility Index (VIX®) moved the most in a single day (http://www.futuresmag.com/2018/02/06/vixs-biggest-one-day-move) in the index's 25-year history. That day, the VIX closed at 37.32, up 20.01 points from the previous day's close of 17.31. The extraordinary move coincided with a steep sell-off in the equity markets with the S&P 500 index falling 4.1%.

The VIX is a mathematical calculation in the form primarily of a weighted sum of mid levels of quoted bids and offers of out-of-the-money put and call options. The CBOE updates the calculation approximately every fifteen seconds (http://www.cboe.com/products/vix-index-volatility/vix-options-and-futures/vix-index/vix-faqs).

Gene Phillips

The weighting of the options is a specific choice that gives a final result after subtraction of a simple term that, in theory, depends only on implied volatility rather than on the level of the equity index. Consequently, the VIX is a market-determined, real-time measure of equity volatility.

The VIX does not trade directly. However, derivatives – including futures and options – directly reference the VIX. Moreover, there are exchange-traded products (ETFs and ETNs) that offer investors exposure to VIX futures.

One noteworthy ETN is the VelocityShares Daily Inverse VIX Short Term ETN (https://sixfigureinvesting.com/2014/05/how-does-xiv-work/) (ticker: XIV), which gives investors the inverse of the daily return on short-term VIX futures. When the VIX rose by 116% on the recent February 5, the short-term futures contracts also rose significantly (by 113%, 87% and 64% for the front-, second- and third-months' contracts, respectively), essentially wiping out investors in XIV. Credit Suisse, XIV's sponsor, announced in short order that it would redeem the notes at large losses and shut down the product (https://www.cnbc.com/2018/02/06/the-obscure-volatility-security-thats-become-the-focus-of-this-sell-off-is-halted-after-an-80-percent-plunge.html).

Various parties have warned of the potential for VIX manipulation. "The VIX has been suspect for at least seven years," former CFTC Commissioner Bart Chilton cautioned in a February 2018 interview on CNBC (https://www.cnbc.com/2018/02/14/ex-cftc-head-bart-chilton-on-whistleblower-vix-manipulation-allegation.html).

Chilton commented in response to a question regarding a whistleblower letter that Zuckerman Law sent to the CFTC and SEC (https://assets.bwbx.io/documents/users/iqjWHBFdfxIU/r8LCxXQ4CfqU/v0) on behalf of an unidentified client. The letter alleges manipulation via the posting of bids and offers on SPX options to affect VIX levels.

On March 9, 2018, Atlantic Trading USA sued unknown "John Does" in a purported class action, alleging manipulation of the settlement price for VIX futures and options (https://www.reuters.com/article/us-cboe-volatility-lawsuit/chicago-firmfiles-lawsuit-alleging-manipulation-of-cboe-volatility-index-idUSKCN1GO2Q9). Atlantic seeks to discover, from third-party (non-defendant) CBOE, the identities of parties to the alleged misconduct.

The Chicago-based trading firm argues that the to-be-identified defendants "caused the monthly final settlement price of expiring VIX contracts to be artificial." They contend further that the defendants did so by "placing manipulative SPX options orders that were intended to cause, and at minimum recklessly caused, artificial VIX contract settlement prices in the expiring contracts." (See *Atlantic Trading USA, LLC v. Does 1-100 (http://www.rrbdlaw.com/3872/securities-industry-commentator/).)*

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In our review of these events and of the VIX methodology, we have identified several aspects of VIX that make the index susceptible to both inadvertent and deliberate variations. We enumerate three specific adjustments of VIX calculations to improve accuracy and reduce susceptibility to errors and manipulation. The most important of these adjustments is the addition of "tail calculations" to fill in the contribution to VIX of missing option prices.

The next section of this article explains the current CBOE calculation methodology as an adaptation of prior literature on the topic of variance swaps. In subsequent sections, we will list approximations of the CBOE method and explain how these approximations promote susceptibility to errors and manipulations.

We will also discuss how to remove or limit these approximations, with a special focus on what we label the Put Tail and Call Tail. Lastly, we will give an extended review of numerical results and consequences of the improvements we propose.

Creating Equity Index Variance with Discrete Options

Demeterfi *et. al.* (http://www.emanuelderman.com/media/gs-volatility_swaps.pdf) (hereafter "DDKZ") provided an authoritative and wide-ranging discussion of variance swaps. A key element of this discussion is the creation of a market instrument that gives an investor explicit exposure to the variance of a desired equity or equity index. This instrument is a portfolio of long call and put options over a wide range of strike prices.

More specifically, the ideal (and clearly unattainable) form of this portfolio requires put options with a continuous range of strike prices, from zero to an arbitrary value K_* and call options, with continuous range of strikes from this same K_* to infinity.

One calculates the value of the option portfolio by summing (integrating) the Black-Scholes (https://www.cs.princeton.edu/courses/archive/fall09/cos323/papers/black_scholes73.pdf) option values with uniform volatility σ. With a highly convenient DDKZ-derived weighting of the option positions, the portfolio value Π (F, K_*, σ, r, T) satisfies

$$e^{rT}\Pi = \frac{F - K_*}{K_*} - \log\frac{F}{K_*} + \frac{1}{2}\sigma^2 T \quad . \quad (1a)$$

In equation (1a), T is the common time to expiration (years) of the option positions, r is the risk-free interest rate to time T, and F is the forward price of the stock (incorporating both the risk-free rate and dividend payments, if any).

The "surprise" of equation (1a) is its simplicity. The volatility dependence is entirely in the last term. An investor who desires a pure long or short position on volatility (or, equivalently, variance in this case) need only take a long or short position, respectively, in the portfolio of options, and then hedge the stock price dependence of the first two terms of (1a).

Alternatively, much easier than hedging these terms, the investor may make just a small adjustment daily to the option portfolio to maintain the value K_* equal to the stock forward price F. This minimal re-balancing keeps these linear and logarithmic terms at zero.

We re-write (1a), as follows, to express the result for variance σ^2:

$$\sigma^2 = \frac{2}{T} e^{rT}\Pi - \frac{2}{T}\left(\frac{F - K_*}{K_*} - \log\frac{F}{K_*}\right) \quad . \quad (1b)$$

CBOE Conversion of Theory to VIX

The CBOE (http://www.cboe.com/aboutcboe/annual-reports) created the "VIX" equity volatility index in 1993 (http://www.cboe.com/aboutcboe/history). CBOE's current VIX calculation, as its white paper (http://www.cboe.com/products/vix-index-volatility/vix-options-and-futures/vix-index/the-vix-index-calculation) shows, employs the DDKZ expression of our equation (1b), but casts it in this form:

$$\sigma^2 = \frac{2}{T}\sum_i \frac{\Delta K_i}{K_i^2} e^{rT} Q(K_i) - \frac{1}{T}\left(\frac{F}{K_0} - 1\right)^2 . \quad (2)$$

The symbols σ, T, r and F are common to both (1a-b) and (2) and have equivalent meaning. The K_i are the strike levels of the SPX (S&P 500® Index) options that CBOE includes in the VIX calculation. ΔK_i is essentially the difference between consecutive strike levels. $Q(K_i)$ is the midpoint of the bid and ask quotes on CBOE for the option with strike K_i.

The CBOE K_O is the special strike value K_* of our equation (1). This is the strike value that separates put options (lower values of K) and call options (higher values of K).

A necessary feature of (2) is that it specifies discrete strike levels K_i rather than a continuous range of strikes of the DDKZ analysis. A true continuum of strikes does not exist.

CBOE employs all strike levels of SPX options with eligible expiry and type that have non-zero bids. As another rule (http://www.cboe.com/products/vix-index-volatility/vix-options-and-futures/vix-index/the-vix-index-calculation), if the SPX options of two consecutive strikes have no bids, then the CBOE calculation excludes all put (call) options at lower (higher) strikes when the strikes of these no-bid options are less (greater) than K_O.

It is noteworthy that CBOE calculates the VIX in "real time" and continuously adjusts the K_O. In theory, the K_* of equations (1a-b) may remain fixed. For practical and intuitive reasons, we agree it is helpful to have K_O (equivalent to K_*) be at or near the index forward price F. Yet the relentless recalculation of K_O is a cost.

Approximations in the VIX Calculation

Comparison of equations (1b) and (2) shows several approximations of the CBOE VIX calculation. First, the far-right terms indicate the CBOE replacement, as follows:

$$\frac{2}{T}\left(\frac{F - K_*}{K_*} - \log\frac{F}{K_*}\right) \rightarrow \frac{1}{T}\left(\frac{F}{K_0} - 1\right)^2$$

(Again, just for clarity, the two symbols K_* and K_O have equivalent meaning. We prefer the former notation for the theoretical development, yet defer to the latter when describing the CBOE implementation.) This approximation offers no apparent advantage. One saves essentially nothing in computation time, and, moreover, the calculation loses accuracy as F moves away from K_O.

Since it is challenging to determine precisely when the approximation is acceptably accurate in different situations, it would be better simply not to make the substitution. A study of A. Li, "A Correction to the CBOE VIX Calculation Formula (https://ssrn.com/abstract=2991829)," is relevant to this topic. Li proposes a correction to this same term, but does not identify the remedy that we consider appropriate.

The CBOE embeds two additional approximations in the first term on the right-hand side of (2). The summation of the term $\frac{\Delta K_i}{K_i^2} Q(K_i)$ is the least accurate expression to represent the (continuous) integral implicit in (1a-b) in that it treats the non-linear integrand as piecewise constant. This is a coarse approximation that would ultimately give an accurate estimate of the true integral value if it were possible to increase the density of strike prices indefinitely.

The strike prices, however, are not arbitrary. The number and spacing of the strike prices is not adjustable. Recognizing the handicap that one cannot assume the K_i to be evenly spaced, an improved calculation would make the following substitution:

$$\frac{\Delta K_i}{K_i^2} Q(K_i) \;\; \rightarrow \;\; \left[(K_i - K_{i-1})^{-1} \log \frac{K_i}{K_{i-1}} - (K_{i+1} - K_i)^{-1} \log \frac{K_{i+1}}{K_i} \right] Q(K_i)$$

The improved version on the right treats the $Q(K)$ as varying linearly between grid points K_i, and then integrates $Q(K)/K^2$ exactly between the K_i.

A third approximation of equation (2) relative to (1b) is the treatment of the tails. We label the "Put Tail" as all strike values less than the smallest strike of (2). Similarly, the "Call Tail" consists of all strike values greater than the largest strike of (2).

The CBOE VIX calculation ignores the tails (*i.e.*, it assumes they have zero value). Yet the tails will have some positive value that, if included, would result in a higher SPX index variance. This missing positive value will, in principle, vary widely depending on the circumstances.

Susceptibility to Error and Manipulation

Clearly and generally, when one makes approximations, it is possible that modeled results will differ significantly from true results. Hence, one should make approximations only when a problem is not otherwise solvable or when there exist other specific advantages to the approximations.

We perceive no such advantages for the case of the CBOE VIX calculation. Let us note, however, that we will show later a typical numerical example in which some of these VIX calculation approximations give only very small errors.

Based on this one example, then, it is quite possible that the approximations for many VIX calculations are not harmful. Yet we do expect some calculations could show significant error. What's more, the existence of approximations necessitates this type of analysis to determine situations in which errors are significant.

Our greater concern and interest lies with the possibility of manipulation. It is conceivable that market participants may exploit the VIX calculation's omission of the tails by adding or removing the options that the calculation includes.

Stated differently, participants may post bids or offers tactically on out-of-the-money SPX options to extend or reduce the range of option strikes in the calculation of equation (2). Extending the range gives higher calculated volatility, while reducing the range induces lower calculated volatility.

A recent study by Griffin and Shams (https://papers.ssrn.com/sol3/papers.cfm?abstract_id=2972979) finds evidence of the manipulation of the VIX at futures settlement dates with targeted buying and selling of SPX options. This investigation specifically considers the possibility that "a large buy order" or "an aggressive sell order" will add or remove SPX options from the VIX calculation (Griffin and Shams (https://ssrn.com/abstract=2972979) at footnote 12). Ultimately, Griffin and Shams allege a manipulative impact of 31 basis points in the VIX index for a period spanning the years 2008-2015.

In principle, mitigating the current VIX calculation assumption of ignoring the tails will suppress the susceptibility to manipulation. As the current method stands, adding more option mid-quotes increases the calculated volatility. Even if these new quotes are reflective of what the market should be for the out-of-the-money strikes, the calculation reflects increases in volatility. If, on the other hand, one would remove the assumption of "zero tails," then this manipulation would be ineffective.

In the next section, we show how to remove the "zero tails" assumption. We simply calculate ideally what the tails "should be" and add the amount to the current calculation. Consequently, any manipulation to increase or decrease the range of eligible strike values for SPX options will shrink or enlarge the extent of the tails. Our calculated tail contribution will then fall or rise correspondingly to mute the manipulation.

Fixing the Put and Call Tails

In calculating the VIX from quotes of option prices, let us imagine that the lowest strike price is L and the highest strike price is H. The current CBOE calculation ignores the tails. However, we can perform integrations similar to that in deriving equation (1a) to find the theoretical values of the tails given a known variance σ^2. We will add these tail values to the following contribution:

$$\sum_{K_i=L}^{H} \frac{\Delta K_i}{K_i^2}\, e^{rT}\, Q(K_i)$$

of the CBOE's calculation employing actual option quotes.

The Put Tail is the integral of weighted put option values from strike of zero to L (multiplied by e^{rT}). Similarly, the Call Tail is the integral of weighted call option values from strike of H to infinity (multiplied by e^{rT}). These tail values are

$$\text{Put Tail} \;=\; \frac{F}{L}\,\Phi(U_-) + \left(\sigma\sqrt{T}U_+ - 1\right)\Phi(U_+) + \frac{\sigma\sqrt{T}}{\sqrt{2\pi}}\,e^{-U_+^2/2} \quad ; \quad (3a)$$

$$\text{with } U_- \;=\; \left[\log\left(\frac{L}{F}\right) - \frac{1}{2}\sigma^2 T\right]\Big/\sigma\sqrt{T} \quad ; \quad U_+ = U_- + \sigma\sqrt{T} \quad ; \quad (3b)$$

$$\text{Call Tail} \;=\; \frac{F}{H}\,\Phi(-V_-) + \left(\sigma\sqrt{T}V_+ - 1\right)\Phi(-V_+) - \frac{\sigma\sqrt{T}}{\sqrt{2\pi}}\,e^{-V_+^2/2} \quad ; \quad (4a)$$

$$\text{with } V_- = \left[\log\left(\frac{H}{F}\right) - \frac{1}{2}\sigma^2 T\right]\Big/\sigma\sqrt{T} \quad ; \quad V_+ = V_- + \sigma\sqrt{T} \quad . \quad (4b)$$

In (3a) and (4a), the notation $\Phi(x)$ denotes the normal cumulative distribution function: $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} dt\, e^{-t^2/2}$.

As a helpful observation, imagine we set $L = H$ in equations (3a-b) and (4a-b) above. This case is not realistic, because the two tails would cover the entire range of feasible strike prices from zero to infinity.

The calculation would simply give the theoretical option portfolio value. Yet that is the purpose of our noting this special case. Adding (3a) and (4a) with $L = H = K_*$ gives our earlier, and much simpler, equation (1).

With this procedure of calculating the tails from a “known variance” σ^2, we would need to iterate, since the variance, in fact, is not initially known. Thus, we would adopt a Fixed Point iteration method (http://mathworld.wolfram.com/FixedPoint.html) to incorporate the tails into the VIX volatility calculation, via the following steps:

1. Apply the CBOE expression of equation (2) to generate the first value of σ. (We continue to recommend that the VIX calculation remove the two earlier approximations we identified. For this discussion, though, we refer to the existing CBOE method.)
2. Employ this σ value to calculate the tails of (3a-b) and (4a-b).
3. Multiply this tail contribution of the preceding step by $(2/T)$ and add it to the right-hand side of (2) to generate a new σ value (square root of the left-hand side of (2)).
4. With this new σ value, repeat steps #2 and #3 above until convergence (*e.*, until the σ value stops changing to a desired precision).

Unfortunately, we find in the next section that the Put Tail is not well described by assuming uniform volatility at the VIX. It is necessary to treat the put option volatility as a function of K/F. Instead of (3a) and (3b), then, we define and perform a numerical integration for the Put Tail with the problem formulation below:

$$\text{Put Tail} = \int_0^{L/F} dx\, x^{-2}\left[-\Phi(u_-) + x\Phi(u_+)\right] \quad ; \quad (5a)$$

$$\text{with } u_- = \left[\log(x) - \frac{1}{2}\sigma^2 T\right]/\sigma\sqrt{T} \quad ; \quad u_+ = u_- + \sigma\sqrt{T} \quad ; \quad (5b)$$

$$\text{and } \sigma \rightarrow \sigma(x) \quad . \quad (5c)$$

The numerical calculation is efficient, so it need not impede "real time" updates to the VIX. For the Call Tail, it appears reasonable to maintain the assumption of uniform volatility, as the next section explains.

Continue Reading Part 2 > (http://www.garp.org/#!/risk-intelligence/technology/quant-methods/a1Z1W000003fOTaUAM)

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Numerical Results

To test and demonstrate the results of applying our proposed modifications of the VIX calculation, we refer to the CBOE white paper. As a method of clearly articulating this calculation mechanism, the white paper gives details of the nearly 150 option quotes and strikes for a specific example.

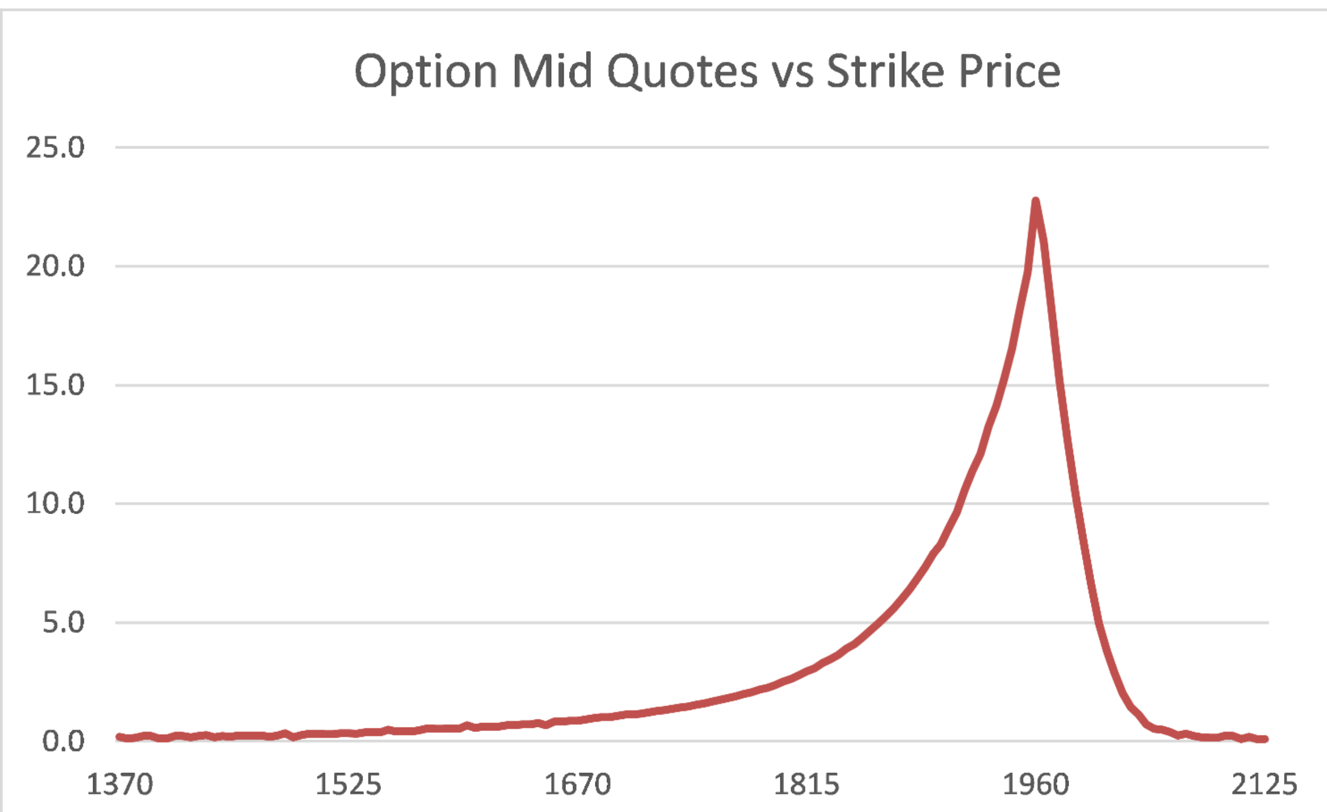
The white paper, moreover, produces numerical values at each step of the calculation to enable readers to reproduce the methods. For this study, we placed all the relevant information for the "Near-Term Strike" in a spreadsheet and confirmed the calculations. This data then forms the basis for our remarks and observations in this section.

Let us note that, on almost all days, the VIX calculation incorporates options that are both somewhat shorter in maturity than 30 days ("near-term") and longer than 30 days ("next-term"). For our purpose, we use only the "near-term." We match precisely the white paper's value for near-term square of the volatility (0.01846292).

The corresponding near-term volatility, which CBOE calls "VIN," is 13.59 (to two decimal places after multiplication by 100). The VIX result of the white paper combines the "next-term" series of option values with the "near-term" values to give an overall VIX of 13.69. But we do not use this combined VIX in our discussion. We focus instead on the VIN.

Figure 1 below shows the prices of all options of the CBOE example as a function of option strike price. The price graph peaks at the strike of 1960 because this is where the transition from put options (at lower strikes) and call options (at higher strikes) occurs. (Referring to equation (2), this transition strike level of 1960 is the K_O of the CBOE VIX calculation.) The forward value of the SPX equity index is 1962.9 – deliberately close to the transition strike value of 1960.

Figure 1: Quotes of SPX Options versus Strike Price



For clarity, we emphasize that the quotes to the left of the peak in Figure 1 pertain to out-of-the-money put options, while the quotes to the right of the peak are for out-of-the-money call options. This figure also defines the range of strike prices (from 1370 to 2125) that the CBOE employed in its VIX calculation.

Observation: VIX is sensitive to the range of strike prices

As we have noted, an approximation of the VIX calculation of equation (2) omits the term $\frac{\Delta K_i}{K_i^2} e^{rT} Q(K_i)$ for all options with insufficient price quotes. Ideally, all options with strike prices K_i, from zero to "infinity," should contribute to (2).

In our CBOE white paper example of this section, the calculation includes no quotes with strike prices less than 1370. Figure 2 (below) shows a portion of available option quotes of the white paper example. (Of the five columns, the strike price is at the far left while the put option bid and offer sit in the fourth and fifth columns, respectively. The second and third columns hold quotes for in-the-money call options, but these are not relevant for our purposes.)

Figure 2: Portion of SPX Options Quotes of the CBOE White Paper

Near-Term Options				
Strike	**Calls**		**Puts**	
	Bid	**Ask**	**Bid**	**Ask**
1350	611.2	614.7	0.05	0.15
1355	606.2	609.7	0.05	0.35
1360	601.2	604.7	0	0.35
1365	596.2	599.7	0	0.35
1370	591.2	594.7	0.05	0.35
1375	586.2	589.7	0.1	0.15
1380	581.2	584.7	0.1	0.2
1385	576.2	579.7	0.1	0.35

By the rules of the calculation, the lowest strike price of included options is 1370. Yet on a different day or at a different time on the same day, this lowest eligible strike price will change.

As one plausible example, imagine that the lowest included option is at the strike value of 1530. We choose this value due to the listed quotes of the CBOE white paper near this value (see Figure 3). The data description of the columns of Figure 3 matches that of the earlier Figure 2.

Figure 3: Portion of SPX Options quotes near the 1530 Strike Price

Near-Term Options				
Strike	**Calls**		**Puts**	
	Bid	**Ask**	**Bid**	**Ask**
1510	451.4	454.9	0.05	0.55
1515	446.4	449.9	0.05	0.55
1520	441.4	445	0.1	0.6
1525	436.4	440	0.3	0.4
1530	431.4	435	0.05	0.6
1535	426.4	430	0.1	0.65
1540	421.4	425	0.1	0.65
1545	416.5	420	0.1	0.65
1550	411.5	415	0.3	0.7
1555	406.5	410.1	0.15	0.7

The bid-side strength for put options is weak in this range. We consider it entirely possible that two consecutive strikes would show no bids at sporadic moments or intervals. Furthermore, a participant with manipulative intent could offer aggressively at these weak points to eliminate bids and thereby shorten the range of options that equation (2) employs in the VIX calculation.

By eliminating the range of options from the 1370 strike to the 1530 strike, the calculated volatility would fall from 13.59% to 13.39%. This is a meaningful deviation for holders of VIX futures and options contracts. We consider this deviation of 20 bps to be plausible, and possibly typical, rather than an extreme case. By a different measure of deviation, for example, Griffin and Shams (https://ssrn.com/abstract=2972979) find a mean disparity of 31 bps.

Observation: Simple calculation of the Put Tail is not meaningful

The weakness of the equation (2) VIX calculation is that it fails to include all possible strike prices from zero to “infinity.” This failing is understandable. When there are no market bids for deep out-of-the-money options, the most evident approximation is to treat the omitted options as having zero or near-zero value.

However, as with any measurement that is both consequential to market players and for which there are clear rules, the approximation invites manipulation. It is conceivable that participants will send bids or offers to attempt to lengthen or shorten the range of eligible option strike prices.

An apparent solution, or at least a mitigant, to this possible manipulation is to add to equation (2) calculations for the Put and Call tails. Thus, if one VIN calculation has a range beginning at the 1370 strike, for example, while a later calculation begins at 1530 strike, then separate calculations for the tails (“zero to 1370” and “zero to 1530,” respectively) could compensate for the 20 bps deviation.

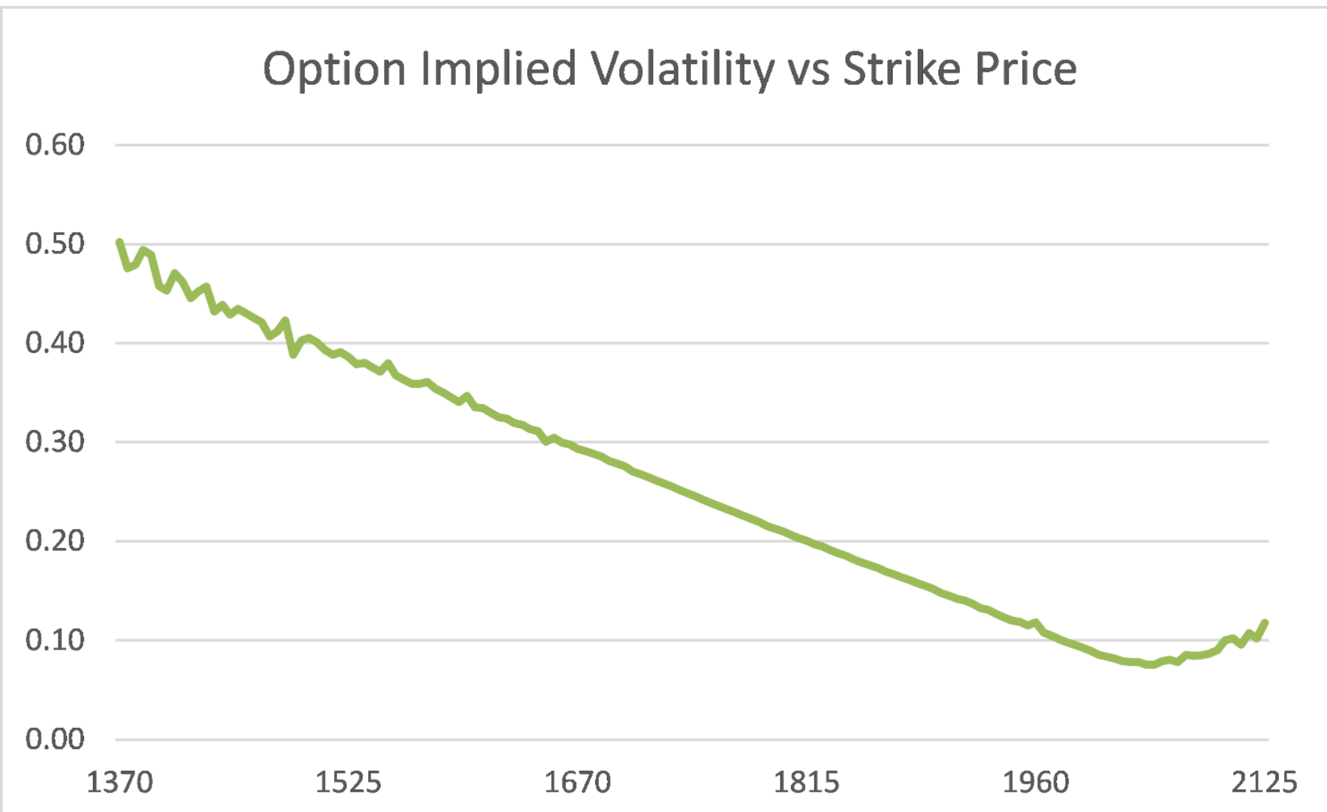
But a first attempt to create this mitigation fails. In applying equation (3a) for the Put Tail for the 1370 strike with the 13.59% volatility, we find a mitigating “correction term” that is exceedingly small (less than one part in 10^{27}). Even for the correction of the tail range to 1530 strike, the calculated correction is just one part in 10^{15}.

A fair conclusion might be that the approximations of omitting these strike ranges (from zero to 1370 and 1370 to 1530) is entirely reasonable. There's a flaw in this logic, however. The VIN result of 13.59% (of this numerical example) is misleading in the sense that the value 13.59% is a blended outcome. The proper volatility of tail ranges – such as zero to 1370 and zero to 1530 – is higher than the "blended" VIN result. Hence, we defer the tail calculation momentarily.

Observation: Volatility skew is critically important to VIX calculation

Just as Figure 1 shows the SPX option quotes as a function of strike price, Figure 4 shows the implied volatility we calculate from the quotes.

Figure 4: Implied Volatility of SPX Options versus Strike Price



Our equations (1a-b), (3a-b), and (4a-b) – as well as the framework of DDKZ and the CBOE VIX calculation of equation (2) – all treat volatility σ as if it is independent of strike price. (DDKZ did include analyses of volatility skew. As they assumed small deviations, their results are not applicable to this case in which the linear increase in put option volatility is large relative to the VIX value.)

Rather than being a mistake in light of the clear strike-dependence in Figure 4, this single-volatility jargon is the convention of the VIX. It's easier to have a single, weighted-average volatility as an index. Yet in any deeper analysis, such as a tail extrapolation, incorporating the strike-dependence of implied volatility is necessary.

For the put options (at strike values less than 1960), the implied volatility is visually linear. The put volatility is ~12% at the K_O of 1960, and rises to ~50% at the end of the range at 1370.

Call options behave differently. Though the call option implied volatility is not constant, neither is it linear or striking in its deviation from its value at K_O. It also bears noting that the range of call options is restricted. The range extends only 10% or so above K_O, whereas the put option range goes 30% below K_O.

Observation: It is necessary to incorporate skew into the Put Tail calculation

Given the clear linear skew of implied volatility for put options, it is not appropriate to apply equation (3a) to calculate the Put Tail. Even if we choose a constant volatility equal to that of the lowest strike option (*i.e.*, ~50% at the 1370 strike), the calculation would arguably underestimate the tail.

This type of procedure – setting a uniform tail volatility based on an illiquid, far out-of-the-money option – might also make the VIX more susceptible to manipulation. Instead, we create a Put Tail calculation that assumes a linear volatility skew. We write again the equations (5a-c) we showed earlier, with a specification now in (5c) for the volatility as a function of x (equivalent to K/F):

$$\text{Put Tail} = \int_0^{L/F} dx\, x^{-2}\left[-\Phi(u_-) + x\Phi(u_+)\right] \quad ; \quad (5a)$$

$$\text{with } u_- = \left[\log(x) - \frac{1}{2}\sigma^2 T\right] / \sigma\sqrt{T} \quad ; \quad u_+ = u_- + \sigma\sqrt{T} \quad ; \quad (5b)$$

$$\text{and } \sigma \rightarrow \sigma(x) = \sigma_0 + \alpha(x-1) \quad . \quad (5c)$$

We observe the volatility parameters σ_0 and α (11.8% and -1.16, respectively) in the data of Figure 4. We then perform numerical integration of (5a-c), with L = 1370, F = 1962.9 and T = 0.0683486, and find that the Put Tail calculation increases the VIN to 13.68% from 13.59% (roughly a 0.7% adjustment).

Separate calculation of the Call Tail does not require this numerical integration. Rather, given the absence of a marked linear skew for call options, we apply equations (4a-b) and find the VIN rises an additional two basis points to 13.70%.

Observation: Inclusion of the Put Tail mitigates VIX sensitivity

Returning to our earlier observation that the VIX calculation is sensitive to the range of strike prices of equation (2), we noted that modifying the lowest strike from 1370 to 1530 decreased the VIN from 13.59% to 13.39%. The procedure of adding the Put Tail to the calculation largely eliminates this sensitivity.

As we noted just above, adding both the Put and Call Tails to the CBOE white paper example gives a VIN of 13.70% (rather than 13.59%). Modifying this example to make 1530, as opposed to 1370, the lowest strike price, our calculated VIN is 13.65% (rather than 13.39%).

Consequently, the tendency of incidental or deliberate changes to the range of eligible options to affect VIX calculations falls greatly when we include the tail contributions. This smaller 5 bps difference (13.65% versus 13.70%) stems from the difference between actual option quotes and the linear skew approximation in the range of 1370 to 1530. The traditional VIX calculation of equation (2) assumes a zero value, rather than the linear skew, for this 1370-1530 range.

Relevance for the VIX of Individual Equities

As the CBOE white paper notes, the CBOE offers VIX indices on individual equities (IBM, Alphabet, Amazon, Apple and Goldman Sachs), as well as on a variety of other equity indices and commodity-based ETFs. Similar to the “conventional” VIX referencing the SPX, the VIX calculation for individual equities employs equation (2). Yet it’s likely that option quotes in single stocks cover a smaller range of strike prices. With a more restricted range, the VIX approximations produce greater errors.

Consider the VIX for the equity of Apple (“VXAPL”). Review of quotes for Apple call and put options at a CBOE website (http://www.cboe.com/delayedquote/quote-table) shows that the put range for purposes of VIX calculation is less than 25% of the equity's forward value. (We reviewed option bids for contracts expiring on the third Friday of the next full month. We searched for consecutive zero bids to determine the minimum put option strike.)

The data for IBM options is comparable. We expect, therefore, that fixing the VIX methodology approximations provides a similar or greater benefit to these single stock VIX variants.

Parting Thoughts

VIX® is a popular, successful and important element of U.S. financial markets. In its disclosure to the public, the CBOE white paper is open and clear in its calculation methodology.

This article highlights the theory behind the VIX and finds three approximations that may leave the VIX prone to error or vulnerable to manipulation: (I) the second term of equation (2) expressing the CBOE method; (II) the implied integral approximation of the first term of equation (2); and (III) the absence of Put Tail and Call Tail contributions in this same first term of equation (2).

We propose that CBOE remove or improve the approximations, and we demonstrate how to do so. The inaccuracy of these approximations is not large, in an absolute sense, for typical scenarios. Yet the small differences we find – almost exclusively in the third approximation pertaining to tail contributions – may be meaningful to the VIX futures and futures options participants.

Beyond the possibility of inadvertent deviations of the VIX, our strong interest also lies with potential manipulation. Fixing the VIX calculation, again especially with regard to the tail contributions, will make the VIX less susceptible to manipulation.

As we describe for the tail calculations, our improved approximation does add new elements to the calculation. We need, for example, a numerical integration for the Put Tail, and suggest a Fixed Point iteration technique to incorporate the Call Tail.

What's more, we implicitly require a numerical estimate of the linear skew parameters for out-of-the-money put options. The improved calculation must have code to implement all three elements. But these additions are not onerous and do not prohibit continuous updates of the VIX.

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