Quant Perspectives

Bernoulli and Behavioral Finance: Both Wrong

The St. Petersburg Paradox – a captivating puzzle created by a famous family of past centuries – remains fun and meaningful today. The behavioral finance community has strong opinions about this clever problem, but risk professionals, as usual, have the best perspective.

By Joe Pimbley

In the history of science and mathematics, there exists the (still) well known Bernoulli family. Numerous members of this Swiss family made brilliant mathematical contributions during the seventeenth and eighteenth centuries. I venture to say that the Bernoullis would be "hell on wheels" in quantitative finance today.

Nicolas Bernoulli's Paradox

As Nicolas' letter of 1713 describes, here's a game (really a financial contract!) that produces a paradox. The "buyer" pays an upfront amount to a "seller" in return for the right to receive a future, initially unknown payment. A sequence of coin tosses determines the seller's payment to the buyer.

If the coin lands "tails" ("T"), then the seller pays \$1 and the contract terminates. If, otherwise, this coin lands "heads" ("H"), then the coin tossing continues until the sequence results eventually in a T. With this first T, the contract terminates and the seller pays $\$2^N$ to the buyer, where N is the number of H recorded in the sequence of coin tosses.

With some thought, one can determine that the probability that the coin toss sequence will produce precisely N "heads" is 2^{-N-1}. For example, this expression gives the probabilities of "zero H" and "one H" as one-half and one-fourth, respectively.

Clearly this feels right, since to get "zero H," the first toss must be a T – and that outcome has probability one-half. To get precisely "one H," the first toss must be H and the second T. That probability is one-fourth.

Since we know both the possible payment values and the probabilities of all these payment values, we should be able to compute the expected payment to the buyer as the sum of the products of values and probabilities. But $\$2^N$ (each payment) multiplied by 2^{-N-1} (each probability) gives one-half for each term. We therefore must add an infinite number of terms, since, in principle, the coin toss could give H for an arbitrarily large number of tosses.

We derive in this way that the expected payment is infinite! (Or some might prefer to say the expected payment does not exist.) Either way, this is a "paradox," because one senses intuitively that there should be a finite "fair value" for the upfront payment. Nobody would make an upfront payment of, say, $3 and feel like they've just achieved immense wealth because they are long a contract of infinite value.

Sensible Solution with Utility

Bernoulli created the concept of "moral expectation" (now known as "utility") as the solution to this paradox. Winning a payment twice as large does not give twice as much "utility" to the recipient, since the postulated utility is sub-linear relative to the payment size. Thus, if one guesses a feasible sub-linear utility function, one may derive a "fair value" upfront payment of the buyer that is finite.

This is something of a fun mathematical game: posit a sub-linear utility function for which one can perform the appropriate infinite summation analytically and find a convergent result. The math works and there's a logic to the approach. We're really not twice as well off if we win $2 billion rather than $1 billion.

Behavioral Finance Assails Bernoulli

Long after Bernoulli's death, the "behavioral finance" ("BF") people came along and attacked him. See, for example, chapter 25 ("Bernoulli's Errors") of Daniel Kahneman's *Thinking, Fast and Slow*.

The BF people like this invention of utility, but they disagree with Bernoulli in making it a simple function of payment amount or current wealth. It is the BF crowd that criticizes the economists for the common assumption that "people and markets are rational." The BF perspective is that utility is important and that it often incorporates irrational biases, such as greater pain for a loss than pleasure for a gain of equal size.

Unlike Bernoulli, to my knowledge, behavioral finance doesn't actually propose a tangible solution to this "St. Petersburg Paradox." It simply criticizes Bernoulli.

By "tangible solution," I mean something that both real people and mathematicians understand and trust. If anybody knows of a clear behavioral finance solution, please contact me!

Risk Professionals Have a Better Answer

Quantitative finance provides a much better answer. Imagine that we treat Bernoulli's game like a financial contract. Dealers would make markets, take both sides of trades, hedge open exposures, create prop desks and hire salespeople for this new "St. Petersburg derivative." We'd also have third-party software and rogue traders.

We don't need or want "utility" to price derivatives. Instead, the astute risk manager would identify quickly the critical aspect of the St. Petersburg trade. It's counterparty credit risk!

The proper valuation would absolutely not employ an infinite series in which one assumes each potential payment is viable. One must truncate the series at or before the point at which the counterparty could not make payment.

Here's a numerical example. Let's say that we distrust our (unsecured) dealer counterparty for any payment obligation in excess of $100 million. Making the overly simple assumption that we trust any payment less than $100 million completely and *distrust* any payment greater than $100 million, we truncate the infinite series after 26 H tosses. That makes the fair upfront payment equal to $13.5. If instead we trust the dealer counterparty for the higher amount of $1 billion, the infinite series goes to 29 H for an upfront payment of $15.0.

There's plenty of room, of course, for better credit risk analyses than the method of the preceding paragraph. One could also add collateral pledging to mitigate counterparty risk with direct effect on the fair upfront payment.

My larger point is that the St. Petersburg Paradox is a clever problem that joins mathematics and practical human interest. The best perspective is not the purely mathematical observation that an infinite series for an expected value diverges. Neither is it helpful to imagine that human beings hold complex and conflicting views about gains and losses.

Rather, it is the evolved quantitative finance and risk management disciplines that bring most clarity to the Paradox.

Joe Pimbley (FRM) is a financial consultant in his role as Principal of Maxwell Consulting, LLC. His expertise includes enterprise risk management, structured products, derivatives, investment underwriting, training and quantitative modeling. Find Joe's archive of previous GARP columns here.