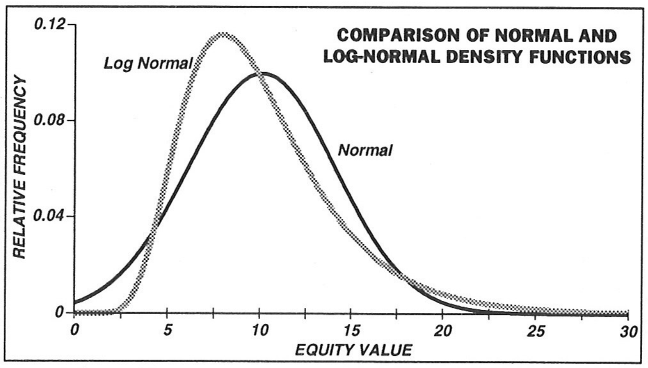
LEARNING CURVE

PROBABILITY DENSITY FUNCTIONS: WHY LOG-NORMAL?

In the pricing or credit risk evaluation of a derivative transaction, one step in a fundamental analysis is the specification of the probability density function (PDF) of the underlying market variable. The value of an option, for example, is the integral of the option payout multiplied by the PDF, where the PDF shows the probability of a given price in the future. Hence, we cannot price an option without this density function.

The PDF for this situation is a curve with non-negative values for all possible equity values from zero to infinity. By definition, the area under the entire PDF curve is one (representing 100% probability) and the probability that the equity value will fall in some range is the area under the PDF curve within the range.

We determine the appropriate PDF by exploiting a surprisingly elegant connection between mathematics and our economic understanding of efficient markets. We first explain the normal and log-normal density functions. The normal density function is the widely recognized bell-shaped curve. The domain (i.e., range of values permitted by the function) of this curve includes all values from minus infinity to plus infinity. Adjustable parameters specify both the center ("mean") and width (proportional to the "standard deviation") of the normal density function. This *Learning Curve* will demonstrate that there is a real theoretical basis for using log-normal distribution in option pricing.



The Central Limit Theorem dictates a key feature of the normal distribution. Though many random variables are not described by the normal density function, a sum of many such independent, non-normal variables is well described by a bell-shaped (that is, normal) curve. Thus, any process in nature or on Wall Street that we can justifiably argue is the sum of independent, random events will possess a normal density function

Next, imagine a quantity (such as an equity or equity index price, a bond price, a foreign exchange rate or an interest rate) that by its nature must be positive. Let us take the (natural) logarithm of this value. This logarithm can be either positive or negative depending on whether the initial positive value is greater than or less than one. When this logarithm has a normal density function, then we consider the original quantity to have a log-normal density function. In other words, a plot of the logarithm of the stock price will appear as a bell-shaped curve (normal density curve). In contrast, a plot of the stock price will be log-normal.

What we gained from this thought exercise other than, perhaps, a headache, is an important consequence of the Central Limit Theorem for log-normal density functions. If we multiply together many independent, positive random variables, then this product will appear to have a log-normal distribution. The logarithm of a product is just the sum of the logarithms of the independent factors. This sum of (independent) logarithms approaches a normal distribution by the Central Limit Theorem. "Undoing" this logarithm then gets us back to a log-normal density function.

Let us say that the stock price at the end of three months will be today's price multiplied by all fractional daily changes. If all fractional market movements are independent, then the equity value density function is log-normal. The mean growth rate and volatility (standard deviation of the logarithm of the distribution) depend on the specific equity and yield curve.

The efficient market view is that last week's run-up in a stock price does not foretell next week's movement. Traders would quickly deflate such perceived trends. Hence, the log-normal density prescription is the mathematical translation of market efficiency.

In the graph, we compare normal and log-normal density functions that might represent the evolution of an equity with an initial value of 10. The log-normal density, which is the appropriate choice, is the more asymmetric of the two curves. The two curves have nearly identical means.

At the risk of belaboring the point, one often hears a justification for the log-normal density as its restriction to positive values which is obviously suitable for many market variables. But other distributions satisfy this positivity restriction as well. It is the market efficiency argument that is most compelling.

There are certainly many situations in which one must apply "log-normality" with trepidation or not at all. Major currency foreign exchange rates, for example, may be subject to Central Bank intervention (especially after an extended movement in either direction). Or, the value of a barrier option depends on a market variable PDF being subject to "boundaries." Hence, the log-normal density, even if appropriate for the "boundary-less" market, fails to provide a complete framework for valuing barrier options.

Swap and bond value movements, neither of which are log-normal processes, are subsets of the broader problem of the stochastic yield curve evolution. Though something of a misnomer, the term "mean reversion" depicts the log-normal model's failure to incorporate the self-influence (through the state of the economy) of extreme interest rate levels. There is no single, universal methodology for tracking the probability density function of future interest rate levels.

This week's Learning Curve was written by ***Joseph Pimbley****, senior analyst-structured finance at* ***Moody's Investors Service****.*