# Mathematical Finance

**Models, Simulation, Today's Pressing Problem**

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## *Outline*

- First Remarks
- Mathematical Finance
- Simulation
- Role of Models
- Valuation (with a story)
- Pressing Problem

- OR methods highly valuable (modeling, optimization, simulation, data analysis, code)
- "Laws of People" – not "Laws of Nature"
- Tremendous diversity
- "People Challenge"

# Stocks

- "Stock" – "Common Stock" – "Equity"
- Incorporated companies, ownership
- No promise of any return of investment, though you can "vote your shares"
- Secondary market – very important!

# Bonds

- "Loans" – "Notes" – "Bills" – "Receivables" – "Debt"
- Companies, gov't entities, consumer
- Borrower pays all interest and principal OR …. not -> "default risk" – "credit risk"
- Much more Debt than Equity

**Video? https://youtu.be/-SoAW2YDx08**

# Derivatives

- "Financial Contract" between two parties
- Value can be *derived* from market variables
- Valuation is a "big deal" – the idea that we calculate a value rather than depend on the market for "price discovery"
- Critical to distinguish "market variable" from "derivative"

# Equity Forward

- Buy Apple stock today for $100
- OR, execute Forward Contract to buy the stock 3 months from today
- Apple stock price today ($100) is a market variable – no math model is necessary
- What should be the "Forward Price?"

# Equity Forward


Buy today
for $100

OR

Agree today to pay $F
for Apple 3 months
forward

**What should $F be? $100?**
**Why would anybody do this trade?**

# Equity Forward


Agree today to pay $F for Apple 3 months forward

**Answer: Think of the dealer!**
**To eliminate risk, the dealer hedges its "forward sale" by buying and holding the Apple stock TODAY. At 3 months, the dealer delivers to the buyer at the agreed $F.**

**So, what is $F?**

# Equity Forward


Agree today to pay $F for Apple 3 months forward

Dealer borrows $100 today, pays interest rate r, and buys the stock

Dealer receives $F, delivers stock, and repays $100 + interest

**Dealer net revenue: $F - $100 (1 + r t)**

**=> $F = S\,(1+rt)$ or $F = S\,\exp(rt)$**

# Equity Forward


- Hedge concept is critical
- No dependence on "opinion" of Apple
- Interest rate for dealer borrowing matters
- Several simplifying assumptions ...

## Black-Scholes and Equity Call Option

Owner of "call option" has the right to buy the underlying stock at a given "strike price" at the "expiration date"

Example: Option to buy Apple stock (current value $100) in 3 months at strike price of $110

How to figure that out? There's no obvious hedge for a dealer.

F. Black and M. Scholes, "The Pricing of Options and Corporate Liabilities," *J. Political Economy*, 637-54, 1973.

## Black-Scholes and Equity Call Option

The value of the dealer's portfolio of the long stock position and short option $P(x,t)$ is

$$P(x,t) \;=\; Nx - V(x,t) \quad (1)\,,$$

where $V(x,t)$ remains unknown. Consider the change in portfolio value $\Delta P$ after the time increment $\Delta t$. This value changes both due to change in time $\Delta t$ and to change in the equity value $\Delta x$. We find (approximately)

$$\Delta P \;=\; N\Delta x - V_t\Delta t - V_x\Delta x - \tfrac{1}{2}\,V_{xx}(\Delta x)^2 \quad (2) \quad .$$

Since the dealer chooses $N$ to best hedge his/her position, it is sensible (and mathematically highly convenient) to choose $N = V_x(x,t)$. Further, to the order of $\Delta t$, the expected value of $(\Delta x)^2$ is $\sigma^2 x^2 \Delta t$ where $\sigma$ is the volatility of the equity and we, following Black-Scholes, have made the industry-standard assumption that the equity price follows a log-normal stochastic process.[10] With these substitutions, we write the change in dealer position value as

$$\Delta P \;=\; -\left(V_t + \frac{\sigma^2 x^2}{2}\,V_{xx}\right)\Delta t \quad (3) \quad .$$

## Black-Scholes and Equity Call Option

$$\Delta P \; = \; -\left(V_t + \frac{\sigma^2 x^2}{2} V_{xx}\right) \Delta t \quad (3) \quad .$$

Next comes a critical observation to which we shall return later. By construction, the dealer's position is riskless over this period $\Delta t$. We don't yet know $V(x,t)$ or the partial derivatives of equation (3), but the Black-Scholes analysis claims that the portfolio value should grow at the risk-free rate since the position itself has no risk. If this risk-free rate is *r*, then, this growth rule suggests

that $\Delta P = rP\Delta t$. Plugging this value for $\Delta P$ into equation (3), invoking $P = xV_x - V$, canceling the factor $\Delta t$, and rearranging terms, we get

$$V_t + \frac{\sigma^2 x^2}{2} V_{xx} + rxV_x - rV \; = \; 0 \quad (4) \quad .$$

## Relevance of the Risk-Free Rate of Interest

- "Risk-free rate" is ambiguous and misused – much like "hedge fund," "liquidity," "arbitrage" and some other terms
- Correct concept is "dealer funding cost"
- In practice, this confusion has been of only minor concern
- Bigger impact among non-practitioners

# Irrelevance of the Equity Expected Appreciation Rate

This equity appreciation rate μ is present in the mathematical formulation in the postulated log-normal stochastic behavior of the equity price in which ε is a standard normal variate $N(0,1)$:

$$\Delta x \;=\; \mu\, x\, \Delta t \;+\; \varepsilon\, \sigma\, x \sqrt{\Delta t} \qquad (6) \quad .$$

Black - Scholes

$$V_t + \frac{\sigma^2 x^2}{2} V_{xx} + r x V_x - rV \;=\; 0 \qquad (4)$$

$$\Delta P \;=\; N\Delta x - V_t \Delta t - V_x \Delta x - \frac{1}{2} V_{xx} (\Delta x)^2 \qquad (2)$$

**Reason for "disappearance" of $\mu$**

## Cox-Ross and the Discovery of Risk Neutrality

- Due to dealer hedge, equity appreciation rate $\mu$ is irrelevant
- Thus, we can choose $\mu$ to be "anything"
- Choice of $\mu$ = "dealer funding rate" simplifies the math => no need to solve PDE explicitly
- This is "Risk Neutrality"

**J. C. Cox and S. A. Ross, "The Valuation of Options for Alternative Stochastic Processes,"** ***J. Financial Economics*****, 3, 145-66, 1976.**

## Cox-Ross and the Discovery of Risk Neutrality

value without formulating and solving the partial differential equation (4). The value of the (call) option at expiration time $T$ is $max(u-K,0)$ where $K$ is the option strike price and $u$ is the equity value prevailing at time $T$. Since we do not know at time $t<T$ what the stock price will be at time $T$, we must think in terms of the probability density function $f(u,x)$ in which the notation makes clear that the density function for $u$ depends on the current stock price $x$. At time $t$, then

$$\text{Expected value of option at expiration} \;=\; \int_0^\infty du\, max(u-K,0)\, f(u,x) \qquad (7)\,.$$

The function $f(u,x)$ incorporates the assignment of $\mu=\hat{r}$. Since the expected value of $V(x,t)$ appreciates at the rate $\hat{r}$, the relation between the option value and the expected value at expiration in (7) is

$$V(x,t)\, e^{\hat{r}(T-t)} \;=\; \int_0^\infty du\, max(u-K,0)\, f(u,x) \quad \text{or, more directly,}$$

$$V(x,t) \;=\; e^{-\hat{r}(T-t)} \int_0^\infty du\, max(u-K,0)\, f(u,x) \qquad (8)\;.$$

## Harrison-Pliska and the Advent of Martingale Measure

- Risk Neutrality implies a Martingale representation property ("risk-neutral measure")
- "Hedge" => "market completeness"
- "Dealer funding" => "self-financing trading strategy" and "short sale of a riskless bond"
- Appears to misunderstand dealer funding cost

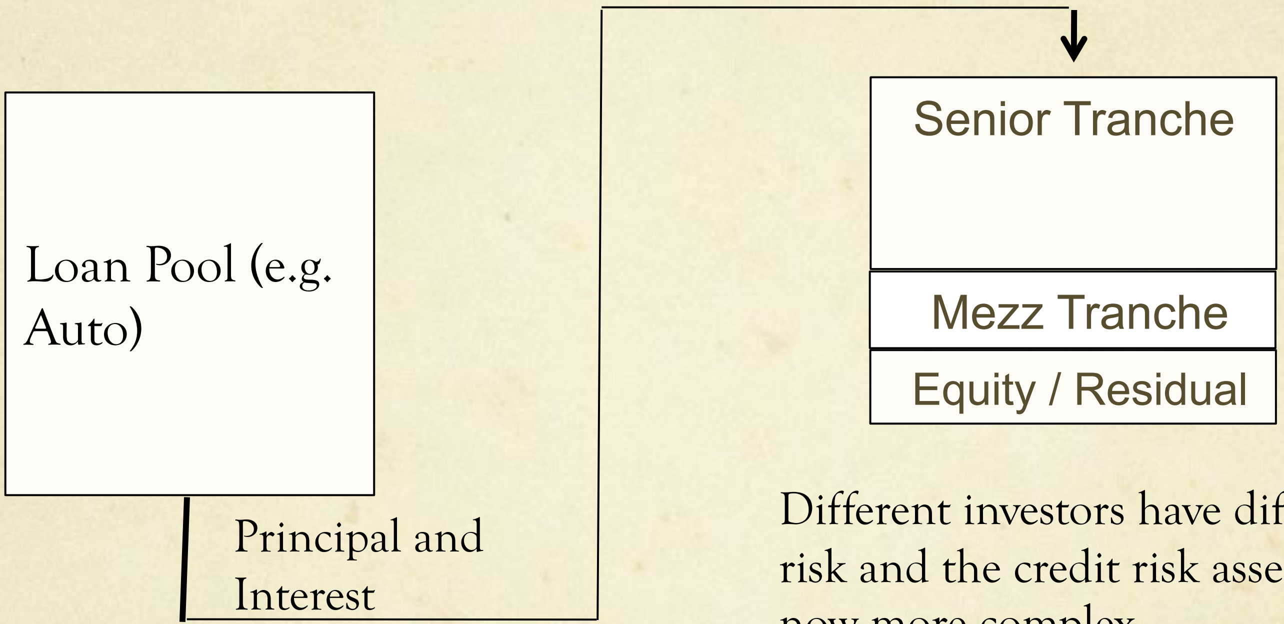
**J. M. Harrison and S. R. Pliska, "Martingales and Stochastic Integrals in the Theory of Continuous Trading,"** ***Stochastic Processes and Their Applications*****, 11, 215-60, 1981.**

## Fundamental Principle for Derivative Valuation

- Hedge Construction
- More broadly, one must understand how the "dealer business" works to derive values
- Many trades are "near-derivatives" in the sense that the proper hedge does not exist – models may be "guides" but risk of error is high

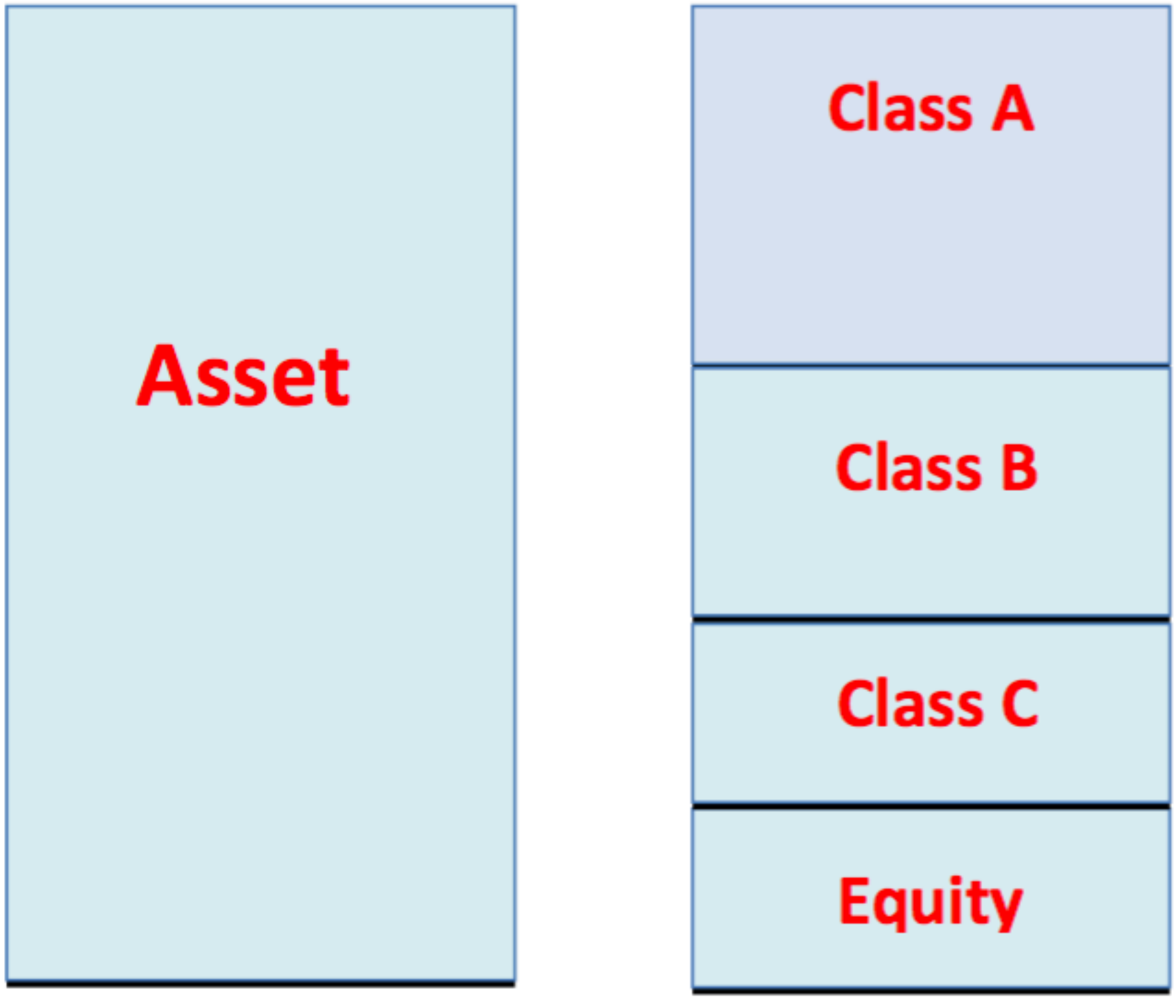
- Numerical methods for problem solving and data analysis
- *Monte Carlo* simulation due to inherent stochastic behavior of markets ("market efficiency" is controversial and a paradox)
- Simulations are "valuation type" or "risk type" (estimate of a probability density function for a portfolio of investments or positions)
- Examples: Structured Finance Cash Flow and Correlated Defaults of a Bond Portfolio

## Structured Finance Cash Flows




Different investors have different risk and the credit risk assessment is now more complex

**Figure 2:** Structured Finance Balance Sheet



## Cash Flow Simulation Outline

1. Set initial conditions (assets, liabilities, terms)
2. Take a step forward in time
3. Change yield curve and other market variables
4. Check for asset defaults
5. Record scheduled asset payments
6. Make scheduled liability payments
7. Return to step 2 if haven't reached maturity
8. Record the loss to investors, if any
9. Return to step 1 if haven't finished all simulations
10. Estimate probability density functions for losses to each tranche

**Write code while reading the legal documents**

## Correlated Defaults

- "Holy Grail" problem
- Specify a default probability for a bond, but a portfolio of bonds has an unknown correlation / interaction
- Correlation is such a helpful "picture" – but it's likely wrong
- Often think in terms of normally distributed variables because they're "easy to correlate"

# Correlated Defaults

$$\hat{t} = t_o \frac{\log\left[1-\Phi(\varepsilon)\right]}{\log\left(1-p_o\right)} \quad (11) \quad .$$

The matrix $A$ describes the linear transformation from independent normal random variables to correlated normal variables. We must choose the lower triangular matrix $A$ to give all desired correlations:

$$A\,A^{+} = R \quad (12) \quad .$$

with $A^{+}$ the Hermitian adjoint of $A$ and $R$ the "correlation matrix." This matrix $R$ is known (or assumed). Cholesky decomposition is the numerical algorithm that finds the matrix $A$ from this relationship above.[22]

- Models are “everywhere” in finance
- Framework of rules, relationships and calculation steps to convert input to output
- Means of converting data to summaries and decisions
- Range from simple to complex
- “All models are wrong, some models are useful” - G. E. P. Box

- Wrong but useful, we all know that
- Even in Laws of Nature, there is friction, linearity assumption, "harmonic oscillator"
- Much less "valid theory" in financial world

## Black-Scholes Option Pricing

- Wrong but useful
- Many small issues: interest rate; dividends; continuous vs daily; "friction;" others
- More important – the geometric Brownian motion is not right
- Most important – it just doesn't work, the ("implied") volatility cannot be determined from historical data

## Black-Scholes Option Pricing

- Wrong, *options are not even derivatives*, so why useful?
- Black-Scholes is a language and thought process for options of all types
- Indispensable aid for estimating value and risk and gaining judgment of option behavior

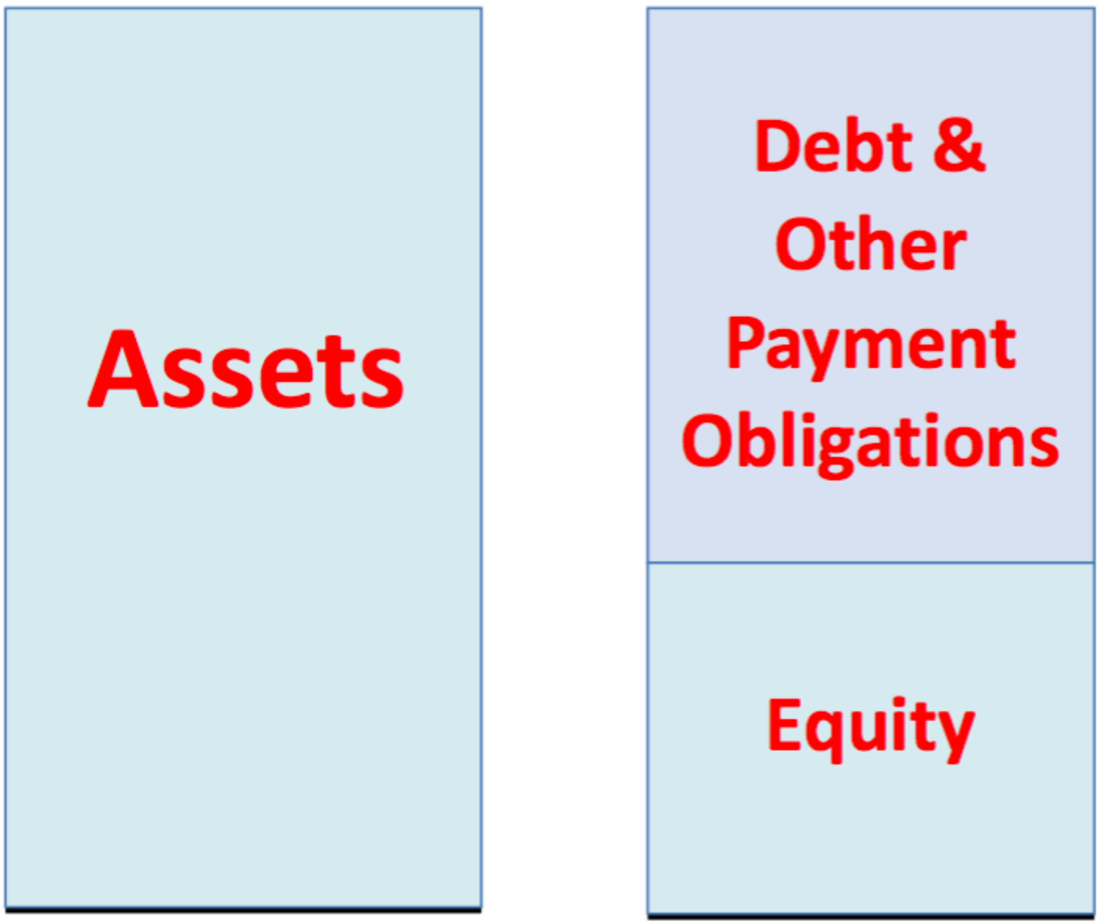
## Merton Debt Default

- Wrong but useful
- "Derivative" of Black-Scholes
- Draw a balance sheet to show relationship of firm equity (lots of good data) to firm debt
- "Equity is a call option on the firm assets with strike price equal to firm debt"

**R. C. Merton, "On the Pricing of Corporate Debt: The Risk Structure of Interest Rates,"** ***J. Finance*****, 29(2), 449-70, 1974.**

## Merton Debt Default

**Figure 3**: Balance Sheet for the Merton Model



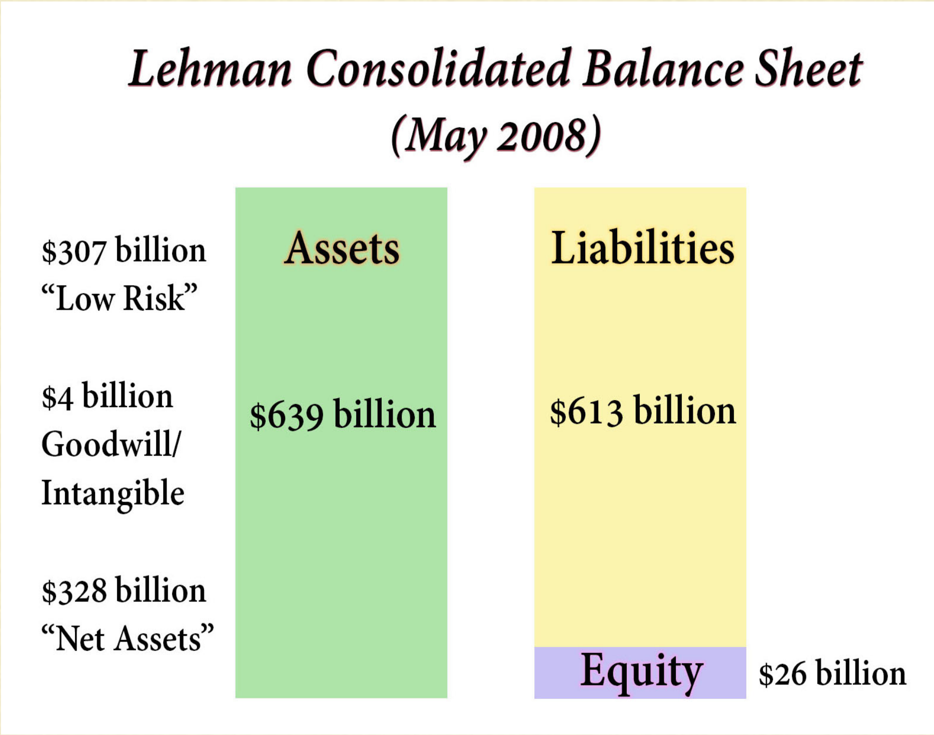
## Merton Debt Default

- Results are wrong
- Unlike Black-Scholes, not even a good estimate
- Initial premise is mis-use of Black-Scholes
- But Merton is useful! Picture and thought process of having asset value and risk drive the analysis of corporate and structured entities is dominant.

## Bank Capital Adequacy

- Wrong and NOT useful – also the greatest global modeling project in finance
- Bank regulators tell their banks how much equity they must have – "adequate capital"
- Ill-posed problem since there is no stated criterion such as "< 1% probability of taxpayer bailout"
- Both the banks and the regulators create the models and generate results – both have incentives and constraints

## BANKS: Fail Frequently – RISKY !




**Huge Leverage – Typical of Banks and Similar Financial Entities**

## Building Useful Models

- Value of models relies on *intent* of builders and users
- Goal is to *seek truth*
- Principle of "good faith" of builders, users, management, regulators
- Beyond "good faith," must realize that the best uses of models are (i) the learning, intuition, and judgment one develops and (ii) the exercise of the firm's data for quality and completeness.

- Deceptively simple in concept
- At what price can one sell an asset?
- Examples
  - Apple stock (highly liquid in small amounts)
  - Interest rate swap (liquid among dealers)
  - Typical bond of small municipality (trades rarely)
  - "Bespoke derivative" (never trades)
- Models have paramount role in some cases
- Valuation results critical in many situations (firm solvency, margin calls, reported profits / losses)

## Story: Mathematical Finance Wins Hundreds of Millions of Dollars

- Longstanding model concept is "LIBOR discounting"
- It's an approximation – ignores pledging of collateral to lower dealer funding cost. For typical dealer book, the valuation impact is very small.
- Would be expensive and complex to change
- Due to aspects of GFC, isolated groups get the idea to liquidate selected positions to create large gain by switching valuation method.

## Misuse of Models

- Absence of "Good Faith"
  - Trading desks pick models to win trades
  - Bank regulators use models to "promote confidence"
  - Gov't agency imposes biased model to enable greater lending to the detriment of borrower protection
  - Unwarranted adjustments to internal risk models at Lehman and other banks
  - Rating agency modifies parameters to cover key error
- Much of this "wrong behavior" is intentional, but there are gray areas
- Math models carry imprimatur of objectivity and impartiality – many people exploit this presumption

## Solution to the Misuse of Models

- Model builders must act in "good faith" AND declare approximations and uncertainties
- Emphasize that models are useful for (i) learning and intuition, (ii) reviewing data and testing its completeness and quality
- Then be prepared to answer "what good are models?"
- Do not impose requirements to build and use specific models

**Joe Pimbley** is Principal of Maxwell Consulting, providing expertise in financial risk management as well as modeling, risk, and valuation analysis for financial asset types including structured products, derivatives, currencies, and debt of corporate, financial, municipal, and sovereign entities. His experience includes leadership of business groups, information technology, enterprise risk management, and quantitative modeling teams. In a prominent consulting engagement from 2009 to 2010, Mr. Pimbley served as a lead investigator for the Examiner appointed by the Lehman bankruptcy court to resolve numerous issues pertaining to history's largest bankruptcy. His work yielded important findings for funding, leverage, collateral, liquidity, and valuation challenges that led to the Lehman bankruptcy. Prior to his 22 years in the financial world, Mr. Pimbley established a 13-year career in physics, mathematics, and engineering. With a PhD in theoretical physics, he researched and developed new microelectronic (semiconductor) devices and taught courses as an assistant professor in mathematical modeling, numerical computing, probability, statistics, advanced calculus, differential equations, and complex analysis.