The Lesser Meaning of Risk Neutrality

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I. Introduction

The advent of credit derivatives will change irreversibly the banking industry. Credit derivatives permit banks to manage and mitigate credit risk in their loan portfolios with no detriment to their borrower (client) relationships. The market for credit derivatives is growing and will continue to grow briskly for the next decade until all major banks incorporate fully this new credit risk management capability.

Somewhat surprisingly, there is still confusion within the financial industry regarding the valuation of all credit derivative instruments. In particular, we focus on the credit default swap which is the most prevalent and basic credit derivative. Roughly speaking, one widely accepted technique to determine the market credit default swap premium is to set this premium to the spread between the yield of a risky bond of the reference entity and the risk-free rate of the same maturity.

Proponents of this "spread to Treasuries" thought process for credit default swap pricing employ a risk neutrality argument that we describe within this article. An investor who purchases a Treasury security (USD currency) has "zero credit risk". If this investor purchases instead a corporate bond, the extra yield (*i.e.*, the "yield spread" or "credit spread") is the additional compensation the investor earns for bearing the credit risk of the corporate borrower. Hence, it seems both sensible and intuitive that the default swap premium to transfer this same credit risk should be approximately equal to this yield spread to the Treasury curve.

The result is clearly wrong, though, in that it differs markedly from premium levels in the existing credit default swap market. There is also a simple hedge argument that demonstrates why the credit default swap premium should coincide more or less with the bond yield spread to the LIBOR curve, as one indeed observes, rather than with the spread to the Treasury curve. Most of the models in the finance literature are of the "spread to Treasury" variety and, hence, show no agreement with the market. A plurality of practitioners, on the other hand, use some form of "spread to LIBOR" model due, if nothing else, to the superior replication of market levels.

It is both important and instructive to dig deeper to understand why two competing modeling approaches give such different results for credit derivative valuation. We find that the manner in which many analysts apply the concept of "risk neutrality" to *any* derivative contract is mistaken. This discussion identifies two observations and one opinion. First, the risk-free rate is irrelevant to derivative pricing (excluding derivatives in which the underlying market instrument is itself a risk-free security or yield). It is LIBOR - which represents a dealer funding cost - that is the proper interest rate for derivative pricing.

Second, what the industry calls "risk neutrality" exists only if a dealer can effectively hedge the derivative product in question. If a hedge does not exist, risk neutrality is absent. Finally, we believe analysts should not consider risk neutrality to be a fundamental principle of finance. Rather, it is more like a very useful trick for which "risk equality" would be a better name. There is no "world", fictitious or otherwise, in which investors demand no compensation for risk. There is only the "trick" that the expected derivative instrument return will equal the dealer funding cost when one forces the expected return of the underlying market variable to be equal to this same funding cost. To some extent, the distinction is semantic. Our goal in raising this point is to discourage analysts from speculating on the behavior of investors in a risk-neutral world rather than simply constructing the relevant hedge.

These observations and opinion may be immediately evident to some readers and create no controversy. But with the example of credit derivative valuation, many finance experts fail to recognize the primacy of LIBOR[*] over the risk-free rate and do not incorporate the dealer's hedge in their pricing models.

Black and Scholes created the option pricing model that is the foundation of this field. Cox and Ross first enunciated the concept of risk neutrality. Harrison and Pliska introduced advanced mathematical techniques to describe and deploy risk neutrality more rigorously. We discuss all of these historic research efforts in sections II, III, and IV, respectively. These sections support our contentions that the risk-free rate is irrelevant and that one must show existence of the cash market hedge in order to price a derivative instrument.

[*] As this article will make clear, we ascribe great importance to the dealer funding cost. Since the senior, unsecured funding cost for many dealers is close to LIBOR of the appropriate tenor, we often use "LIBOR" as a shorter name for "dealer funding cost of the hedged portfolio".

Section V argues our opinion that the industry has accorded too much meaning to risk neutrality. The useful concept is more akin to a trick with the name of "risk equality" since one may solve for derivative valuation with the fiction that the derivative and the cash security have expected return equal to the dealer's funding cost.

Section VI gives several examples of derivative pricing and the utility of the conventional risk neutrality. The credit default swap is the last case. We define this instrument and show in a simplified way how many analysts apply risk neutrality to its valuation. Section VII summarizes this article.

II. Black-Scholes and the Hedged Portfolio

We consider the original article of Fischer Black and Myron Scholes[1] to be one of the best theoretical research efforts we've ever studied. The central concept is that one must consider the dealer's hedge for a derivative transaction, in this case an equity option, in order to determine "rational value" for the derivative. The hedge is integral to the derivation of the value of the derivative trade.

We give here a brief review of the derivation of the Black-Scholes partial differential equation for pricing equity options. We then discuss the inference of "risk neutrality" from this analysis and argue that the significance of this principle is limited. Finally, we note that the "risk-free" interest rate is not the relevant interest rate for the Black-Scholes analysis.

Derivation of the Black-Scholes Partial Differential Equation

Our problem is to find the value $V(x,t)$ of a call option on one share of an equity with value x at time t. The equity pays no dividends. The option expires at time T with $T > t$ and we know the expiration value of the option is $V(x,T) = \max(0, x - S)$ where S is the option strike price. A natural approach to a solution for $V(x,t)$ is to determine somehow the probability density function for the equity value at time T, compute the option expected value from the pay-off function $\max(0, x - S)$, and then discount this value to time t. This approach failed prior to Black-Scholes. The hardest steps are the discovery of the expected equity growth rate and the proper (risky) discount factor. Nobody knew then, or knows now, how to discount risky cashflows or specify a risky asset's expected growth rate.

Instead, the Black-Scholes approach requires the dealer to hedge its position in the equity option and then focuses on what happens to the value of this hedged position over a short time increment Δt.* I generally think of this

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* For this discussion, we consider the position of a dealer that has sold the equity call option to an investor.

Δt as being one business day to help myself make a picture for the analysis. The dealer must purchase *N* shares of the equity to hedge its (short) call option obligation to the investor (option buyer). Since $V(x,t)$ is the value of the option to buy one share of equity, *N* is a fractional value less than one.

The value of the dealer's portfolio of the long stock position and short option $P(x,t)$ is

$$P(x,t) = N\,x - V(x,t) \quad \text{(II-1)}$$

where $V(x,t)$ remains unknown. Consider the change in portfolio value ΔP after the time increment Δt. This value changes both due to change in time Δt and to change in the equity value Δx. We find (approximately)

$$\Delta P = N\Delta x - V_t\Delta t - V_x\Delta x - \frac{1}{2}V_{xx}(\Delta x)^2 \quad \text{(II-2)} \qquad .$$

Since the dealer chooses *N* to best hedge his/her position, it is sensible (and mathematically highly convenient) to choose $N = V_x(x,t)$. Further, to the order of Δt, the expected value of $(\Delta x)^2$ is $\sigma^2 x^2 \Delta t$ where σ is the volatility of the equity and we've made the standard assumption that the equity price follows a log-normal stochastic process. With these substitutions, we write the change in dealer position value as

$$\Delta P = -\left(V_t + \frac{\sigma^2 x^2}{2}V_{xx}\right)\Delta t \quad \text{(II-3)} \qquad .$$

Next comes a critical observation to which we shall return later. By construction, the dealer's position is riskless over this period Δt. We don't yet know $V(x,t)$ or the partial derivatives of equation (II-3), but the Black-Scholes analysis claims that the portfolio value should grow at the risk-free rate since the position itself has no risk. If this risk-free rate is *r*, then, this growth rule suggests that $\Delta P = rP\Delta t$. Plugging this value for ΔP into equation (II-3), invoking $P = xV_x - V$, canceling the factor Δt, and rearranging terms, we get

$$V_t + \frac{\sigma^2 x^2}{2}V_{xx} + rxV_x - rV = 0 \quad \text{(II-4)} \qquad .$$

Equation (II-4) is the celebrated Black-Scholes equation. Solving this equation with the expiration condition $V(x,T) = \max(0, x - S)$ gives the Black-Scholes expression for the call option value which we decline to show here. Equation (II-4) applies to all equity derivative trades that a dealer can hedge with a position in the underlying stock. For example, consider the equity forward which we will discuss in a later section. When the investor buys the equity for forward settlement at time T at the forward price F, the expiration condition is $V(x,T) = x - F$. The solution of (II-4) subject to this auxiliary condition is

$$V(x,t) = x - Fe^{-r(T-t)} \quad \text{(II-5)} \qquad .$$

Black also derived the value of a forward contract in terms of the relevant futures price in this manner.[2]

The Black-Scholes Equation Implies Risk Neutrality ?

Equation (II-4), the Black-Scholes equation, has the risk-free interest rate r as a parameter but excludes the underlying stock's expected drift rate μ. Even without solving (II-4), then, we know that the equity derivative value will be independent of μ. Indeed, the value of the equity forward in (II-5) is not dependent on μ. One must ask why there is no μ-dependence and what this lack of μ-dependence means. One would intuitively expect the value of an equity option or forward to depend on the equity's expected rate of appreciation (which will always exceed the risk-free rate).

This equity appreciation rate μ is present in the mathematical formulation in the postulated stochastic behavior of the equity price:

$$\Delta x = \mu \, x \, \Delta t + \varepsilon \, \sigma \, x \sqrt{\Delta t} \quad \text{(II-6)} \qquad .$$

In equation (II-2), Δx appears as the multiplicand of $(N - V_x)$. To hedge the dealer's position, though, we set $N = V_x$ and thereby eliminated μ and all other elements of Δx. The only other appearance of Δx in (II-2) is in the term containing $(\Delta x)^2$. When we square Δx, though, we retain only the leading term proportional to Δt. The terms in $(\Delta x)^2$ that depend on μ disappear from the mathematical model faster than $O(\Delta t)$ as $\Delta t \to 0$.

The most captivating reason for the absence of μ from the equity derivative pricing equation (II-4) is the first: the dealer's ability to hedge his/her position makes the true equity appreciation rate μ irrelevant. Let's say that again. When the dealer can hedge his/her position, the value of the equity derivative trade is completely independent of the true equity appreciation rate μ. Conversely, if the dealer <u>cannot</u> hedge the position (due to an inability to buy or sell the underlying equity), then the mathematical model that gives us equation (II-4) is wrong. One should then expect that the equity derivative value <u>will</u> depend on the equity appreciation rate μ.

It is the lack of dependence of derivative value on appreciation rate μ that the market declares to be "risk neutrality". Since the correct derivative valuation (when the dealer can hedge the position) does not recognize that the equity appreciation rate μ is greater than the risk-free rate r, the term "risk neutrality" appears at this point quite understandable.

The Black-Scholes Interest Rate is Not the Risk-Free Rate

Let's return to equation (II-1) and the surrounding discussion. An investor purchases an equity call option from a dealer. The dealer purchases N shares of the underlying equity to hedge its position. The value of the dealer's portfolio is (reprising equation (II-1))

$$P(x,t) = N\,x - V(x,t) \quad \text{(II-1)}$$

where $V(x,t)$ is the value of the equity option. The dealer must borrow to purchase the N shares of equity (at a price of x per share). Since the investor paid $V(x,t)$ for the option, the dealer has a net borrowing requirement of $N\,x - V(x,t)$ - which is precisely the portfolio value.

The interest rate the dealer must pay to borrow in the market is $\hat{r}$. We call this value the dealer's "funding cost" or "borrowing cost".* The funding cost $\hat{r}$ is greater than the risk-free rate r. Since the dealer must borrow at $\hat{r}$, it will also expect and demand that the value of the hedged portfolio (II-1) will appreciate at the rate $\hat{r}$ (or greater). We express this portfolio value appreciation as $\Delta P = \hat{r} P \Delta t$ rather than the $\Delta P = r P \Delta t$ prescription we quoted following equation (II-3).

After this substitution of $\Delta P = \hat{r} P \Delta t$ for $\Delta P = r P \Delta t$, $\hat{r}$ becomes the interest rate in all the subsequent mathematics. The risk-free rate r never reappears. If one understands and accepts that the dealer must earn its funding rate $\hat{r}$ rather than the risk-free rate r on its hedged portfolio, then the funding rate is the proper interest rate for Black-Scholes analysis. Throw out the risk-free rate. The risk-free rate is irrelevant.

This irrelevance of the risk-free rate r works in the opposite direction as well. When the investor <u>sells</u> the equity call option <u>to</u> the dealer, the dealer will hedge its position by selling short the underlying equity. To borrow the equity for the short sale, the dealer must invest cash with the equity lender. This equity lender will pay the dealer the lender's (equity collateral-secured) borrowing cost $\hat{r}$. Again, the relevant interest rate is a dealer borrowing rate rather than the risk-free rate r.

The Dealer Borrowing Cost $\hat{r}$ is LIBOR

* In principle, the dealer may post the N equity shares as collateral for the loan in an attempt to lower its borrowing cost.

The inter-bank lending rate is LIBOR (London Inter-bank Offer Rate). While LIBOR represents market assessment of unsecured credit risk for the average financial institution, market participants generally use LIBOR as "the borrowing rate" for dealers without distinguishing secured versus unsecured. (An important exception is repurchase agreements with US Treasury collateral. The "Treasury repo" rate is essentially the risk-free rate by virtue of the collateral quality.) Due primarily to various regulatory treatments, the posting of equity or corporate debt collateral often does not significantly reduce a dealer's borrowing cost. Hence, we are in the habit of equating "dealer borrowing cost $\hat{r}$" and "LIBOR".

But Isn't the Portfolio Risk-Free ?

We argued that the risk-free rate is irrelevant since the dealer must fund the hedged portfolio (in the case in which the investor buys an equity call option from the dealer). Since the dealer must fund at the dealer borrowing cost $\hat{r}$, the value of the risk-free portfolio must grow at this rate as well. Otherwise the dealer would suffer a certain loss.

Yet there's something missing here. By both construction and assumption, the dealer's portfolio of the call option obligation and the N shares of equity is risk-free over the time period Δt. If the portfolio is risk-free, why should it not appreciate at the risk-free rate?

In the currency of the US dollar, US Treasury securities appreciate at the risk-free rate. Dealers will buy Treasuries even though they appreciate at the risk-free rate rather than at the dealer's borrowing cost $\hat{r}$. The dealers can fund Treasury purchases with Treasury repos (in which the dealer pledges its new Treasury securities to the lender) to reduce the dealer borrowing cost to the risk-free rate. It is not possible, though, for the equity option dealer to pledge its hedged portfolio in the same manner. Thus, if the dealer must borrow at $\hat{r}$, it will demand that all investments appreciate at a rate greater than or equal to $\hat{r}$.

A reasonable question would then be whether the dealer could fund the purchase of the risk-free portfolio from its equity rather than its debt. The creation of the hedged portfolio would then be unleveraged in that the dealer would not borrow any funds so that there is no "dealer borrowing cost" $\hat{r}$ to compete with the risk-free rate r. But this is not how the financial industry works. Equity investors of the dealer take significant risk and expect a commensurate return. The "cost of capital" is the rate that equity investors wish to earn. This cost of capital far exceeds both the risk-free rate and the borrowing cost $\hat{r}$. In fact, dealers try to use debt as much as possible rather than equity since the former is less expensive. Typically, a dealer borrows at $\hat{r}$ to fund all trades and then allocates capital (with the associated cost of capital) in an amount appropriate to the risk of the trade to the dealer. Since we assume the dealer has zero risk in the equity option hedge, the dealer would assign no equity capital to the hedge. The cost for a zero-risk transaction, then, is $\hat{r}$.

III. Cox-Ross and the Discovery of Risk Neutrality

Cox and Ross contributed a landmark scholarly article in the dawn of the Black-Scholes era.[3] Their study described stochastic processes intuitively in terms of jumps rather than diffusion alone. They formulated from first principles the Cox-Ross “square root process”. Even more important, Cox and Ross elaborated on both the Black-Scholes concept of the hedged portfolio and the nature of the risk-free rate (which they denoted as the “riskless” rate). Finally, Cox and Ross discovered risk neutrality. This research effort was and remains highly impressive. This section explores the latter two achievements.

Cox and Ross echoed Black and Scholes in stating that equity option valuation requires both the specification of a stochastic process for the underlying equity and the construction of a hedged portfolio. Analysis of the hedged portfolio entails the existence of a risk-free rate “which we will take to be a constant rate at which individuals can borrow and lend freely”. This stipulation is critically important! The “riskless” rate of Cox and Ross is, in fact, a borrowing and lending rate for individual investors. In the preceding section, we argued that the appropriate interest rate for derivative product analysis is the borrowing and lending rate for *dealers* … which we assert is semantically equivalent to *individuals*. The Cox-Ross research is never more specific about the “riskless” rate. It never specifies the true risk-free rate (of US Treasury securities for the US dollar currency) as the relevant “riskless” rate. Our prior conclusion, then, that the dealer funding cost, rather than a true risk-free rate, is the proper interest rate is fully consistent with Cox-Ross.

Let’s move next to the discovery of risk neutrality. Cox and Ross observed from equation (II-4), as did Black and Scholes, that the true equity appreciation rate μ is irrelevant (due to the hedge construction). In a practical sense, this irrelevance of μ is quite helpful. The log-normal stochastic process for the equity requires both this appreciation rate μ and the equity volatility σ. Since the value of μ cannot affect the derivative value, we must only determine the volatility σ. We can then leave this rate μ as undecided or, if we desire, choose any convenient value for μ.

Next comes the revelation. Consider equation (II-1) for the hedged portfolio value for the dealer that has sold an equity option (which we copy here and re-number):

$$P(x,t) = N\,x - V(x,t) \quad \text{(III-1)} \quad .$$

The hedge construction discussion of the last section showed that this portfolio value $P(x,t)$ must grow in time at the rate of the dealer's funding cost $\hat{r}$. The two terms on the right-hand side of (III-1) are stochastic, but the difference in these two terms is not stochastic. When the expected growth rate of $N\,x$ is the general value μ, then the expected growth rate of $V(x,t)$ is not constant in time in that it varies with changing values of $N\,x$ and $V(x,t)$.

But there is one very special case. If we choose μ to be the dealer funding cost $\hat{r}$, then the expected growth rate of $V(x,t)$ is also $\hat{r}$ for all time regardless of $N\,x$ and $V(x,t)$ as long as the dealer maintains the hedged portfolio. That is, since $P(x,t)$ appreciates at the rate $\hat{r}$ (by hedge construction) and the expected value of $N\,x$ appreciates at the rate $\hat{r}$ (after setting $\mu = \hat{r}$), then equation (III-1) implies that the expected value of $V(x,t)$ must appreciate at the rate $\hat{r}$. This observation is the core of Cox-Ross risk neutrality. Since it doesn't matter what assumption one makes for μ, make the convenient choice that $\mu = \hat{r}$. This specific choice gives the very simple expected growth rate of $\hat{r}$ for $V(x,t)$.

One may exploit this property of equation (III-1) to derive the equity option value without formulating and solving the partial differential equation (II-4). The value of the (call) option at expiration time *T* is $\max(x - S, 0)$ where *S* is the option strike price and *x* is the equity value prevailing at time *T*. Since we do not know at time $t < T$ what *x* will be at time *T*, we must think in terms of the probability density function $f(x)$. At time *t*, then

$$\text{Expected value of option at expiration} = \int_0^\infty dx \max(x - S, 0)\, f(x) \quad . \qquad \text{(III-2)}$$

The function $f(x)$ incorporates the assignment of $\mu = \hat{r}$. Since the expected value of $V(x,t)$ appreciates at the rate $\hat{r}$, the relation between the option value and the expected value at expiration in (III-2) is

$$V(x,t)\, e^{\hat{r}(T-t)} = \int_0^\infty dx \max(x - S, 0)\, f(x) \quad \text{or, more directly,}$$

$$V(x,t) = e^{-\hat{r}(T-t)} \int_0^\infty dx \max(x - S, 0)\, f(x) \quad . \qquad \text{(III-3)}$$

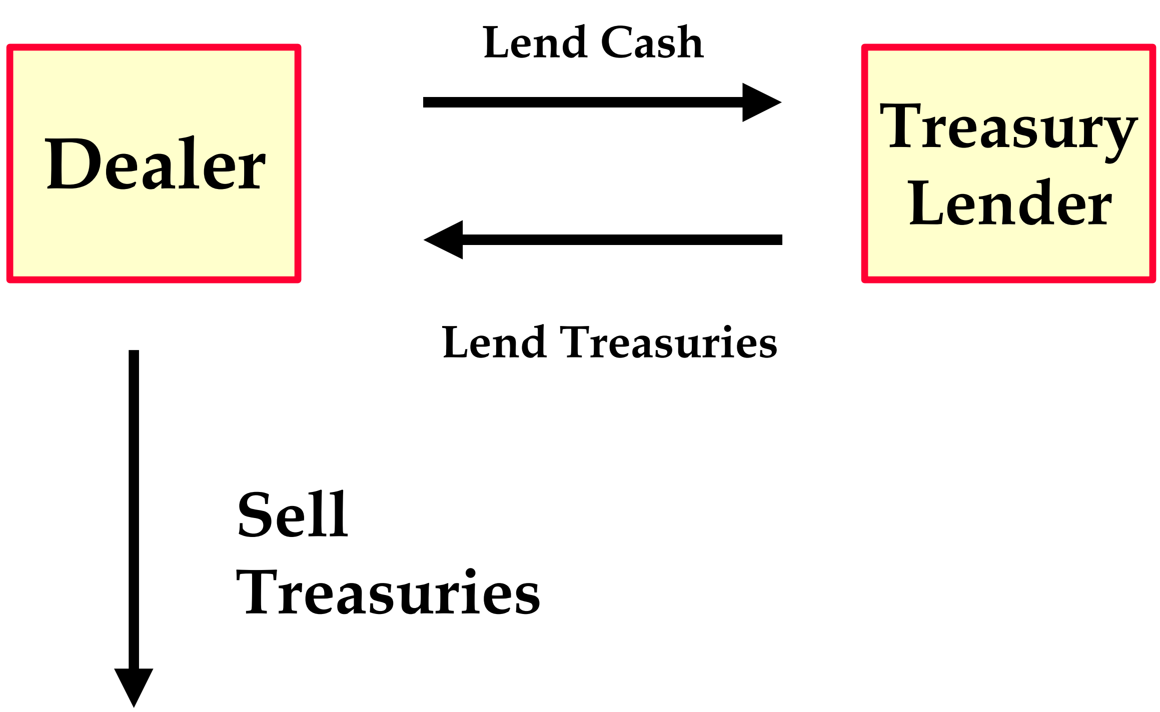
Equation (III-3) is the complete solution to this equity option valuation when we choose a specific probability density function $f(x)$ - such as the log-normal density function - with the constraint that $\mu = \hat{r}$.

IV. Harrison-Pliska and the Advent of Mathematical Finance

Harrison and Pliska re-formulated the equity option problem and the Black-Scholes solution to show that risk neutrality implies a certain martingale representation property.[4] Many, many subsequent finance research efforts have employed probability calculations “in the risk-neutral measure” to create derivative pricing results. That is, Harrison-Pliska conferred a far more esoteric and mathematical meaning to risk neutrality than that of a simple consequence of a hedged position.

This research effort did not argue that a dealer’s hedge is irrelevant. Rather, it changed the language of the discussion. The “hedged position” became a “self-financing trading strategy”. The assumption of “market completeness” replaced the Black-Scholes hedge construction. The portfolio funding requirement evident in the hedge construction exercise devolved into the short sale of a “riskless bond”. Harrison-Pliska explained that “short selling amounts to borrowing … money at the riskless interest rate *r*”.

To the best of our knowledge and experience, this analytical transformation will yield correct results with one critical *caveat*. This Harrison-Pliska assertion that “short selling amounts to borrowing …” is <u>only</u> appropriate if the “riskless interest rate” is the dealer’s own funding cost rather than the true risk-free interest rate (*i.e.*, the US Treasury rate for US dollar borrowing). The article appears to imply - without doing so explicitly - that the proper trading strategy includes a short position in US Treasury debt rather than the dealer’s own debt. Consider this diagram of a (government) Treasury debt short sale:



The dealer executes a short sale of the Treasury securities. That is, it sells securities that it does not own. The dealer borrows the Treasuries from the Treasury Lender. But the dealer must post cash in an amount generally greater than the value of the Treasury securities it borrows. This cash pledge greatly reduces the credit risk of the Treasury Lender to the dealer. The arrangement in the diagram is the familiar "Treasury repurchase agreement".

Thus, it is false to presume that a dealer may sell short government debt securities to effectively fund itself at the true risk-free interest rate. The only debt that a dealer can sell without posting collateral is its own. A dealer that "sells its own debt" is simply funding itself at its own funding cost. The Harrison-Pliska analysis, then, is correct if the "riskless bond" in the self-financing trading strategy is the dealer's own debt.

We have two qualms about the Harrison-Pliska view of derivative pricing that has proliferated so widely. We outlined the first in the preceding paragraphs. Many analysts mistakenly designate the true risk-free (*i.e.*, government debt) interest rate as the "riskless interest rate" for the subsequent analysis.

Our second objection is that the Harrison-Pliska approach replaces the question "does a hedge exist?" with "is the market complete?" The former query is specific and the reply is readily apparent. If the analyst claims that a hedge exists, he/she will describe the hedge. The analyst would be foolish to state that "I assume a hedge exists but I just don't know what it is." The latter inquiry, though, is vague. The analyst who assumes market completeness cannot prove the assumption, so there's a strong tendency to assume completeness when this assumption is wrong.

If one cannot hedge a particular contingent claim, then it is "not attainable" in the words of Harrison-Pliska. Since this claim is not attainable, the market is, therefore, not complete. Hence, if the hedge is non-existent, Harrison and Pliska and their right-minded followers would agree that the market completeness assumption is flawed. In practice, though, many analysts fail to consider the hedge existence at all. Yet it's so much easier and more intuitive to ponder the hedge than the abstract "market completeness"!

V. True Meaning of Risk Neutrality

The Cox-Ross mathematical observation that the option value must appreciate at the rate $\hat{r}$ if the underlying equity appreciates at this same rate greatly simplifies option valuation and seems to provide some additional insight. One essentially "pretends" that the equity appreciates at the rate $\hat{r}$ even though the true (and unknown) appreciation rate μ is greater than $\hat{r}$. Further, one then discounts the expected expiration value at the rate $\hat{r}$ which is not ordinarily the proper discounting method for an uncertain return.

We view this "risk neutrality" observation as a convenient and clever "trick" for deriving some derivative pricing results easily and quickly. The greatest insight we deduce is that it is the presence of the hedge that makes this trick work. Without the hedge, there is no basis for the claim of independence of derivative pricing on the true appreciation rate μ.

But many in the financial community have mistaken "risk neutrality" for a fundamental principle rather than a trick that derives its justification from the existence of a hedge. The distinction, "fundamental principle" or "trick", may be largely semantic. The best example of the mistaken application of "risk neutrality" is in the pricing of credit derivatives. Some practitioners identify "risk-neutral default probabilities", "risk-neutral recovery rates", and "risk-neutral rating transition probabilities" with no explicit reference to a hedged portfolio.[5,6,7,8] This approach, which we discuss as one example in the next section, is baseless.

If nothing else, "risk neutrality" has the wrong name. As we've argued and will show with examples, the relevant interest rate is the dealer funding cost $\hat{r}$ rather than the risk-free rate r. The Cox-Ross construction confirms this view. Since this dealer funding cost $\hat{r}$ is markedly higher than the risk-free rate, the dealer who demands a return of $\hat{r}$ for its hedged portfolio is not at all insensitive to the non-zero risk of the equity. Perhaps a better name would be "risk equality" to describe the trick that the presence of a hedged portfolio permits the analyst to set the appreciation rate of both the underlying equity and the equity option to the market rate $\hat{r}$ of the dealer's debt (*i.e.*, to the dealer's funding cost).

There is no "world" - fictitious or otherwise - in which investors require no premium for the risks they take. There is only a useful trick of "risk equality" that derives its legitimacy from a clearly defined hedged portfolio.

VI. Risk Neutrality is the Wrong Paradigm .... with Examples

The claim that "risk neutrality is the wrong paradigm" is vague and admits little subtlety and nuance. So perhaps the statement is too weak. There do exist situations which we describe here, however, in which the direct application of risk neutrality gives an incorrect result. That's why we label the risk neutrality paradigm as "wrong". But the application of risk neutrality does often give seemingly correct and sensible results. Even in these cases, though, there is confusion regarding the "risk-free rate" since the principle of "risk neutrality" suggests by its name that one should use the US Treasury rate for the USD currency. After reviewing several examples, we conclude that focusing on "hedge construction" gives clearer, simpler, and more uniformly correct solutions for derivative pricing compared to the "risk neutrality" archetype.

Example: Equity Forward

Consider first one of the simplest derivative trades: the equity forward. A dealer and an investor enter into a financial contract in which the investor agrees to pay a fixed price F to purchase 100,000 shares of *Microsoft* (MSFT) stock six months following the effective date of the contract. The investor's motive is to profit from an appreciation of MSFT stock at the six-month expiry without a funded purchase of the equity position. The dealer's motive, of course, is to accommodate the investor and earn a profit.

Pricing this equity forward is equivalent to deriving the appropriate fixed price F for the transaction. While the dealer would like F to be as high as possible and will negotiate with this incentive, we compute F here as the value that gives the dealer precisely zero profit when the dealer hedges its equity risk. Consider this F as the mid-market level. As a simplification, we assume that neither the investor nor the dealer will default on their obligations in this contract so that "counterparty risk" will not impact the forward stock price F.

To derive the forward price F, we construct the hedge the dealer will use to negate its equity risk. At contract execution, the dealer should buy 100,000 shares of MSFT. The dealer will hold the shares until it delivers them to the investor in six months in return for the investor's payment of F. The MSFT stock pays no dividend. The dealer must borrow an amount S equal to 100,000 multiplied by the current MSFT share price to fund the purchase of the hedge.

At the six-month expiration, the dealer must repay its loan. This loan repayment will be $S\left(1+\frac{\hat{r}}{2}\right)$ where $\hat{r}$ is the dealer's six-month borrowing rate. This rate $\hat{r}$ is **not** a risk-free rate. It is a dealer funding rate which we typically take to be LIBOR (the London inter-bank offer rate). Six-month LIBOR is typically 20-50 basis points per annum higher than the six-month US Treasury rate. Hence, the difference is substantial.

Since the dealer must pay $S\left(1+\frac{\hat{r}}{2}\right)$ at expiration, the value of F must also be $S\left(1+\frac{\hat{r}}{2}\right)$ for the dealer to have precisely zero profit. Thus, the solution is

$$F = S\left(1+\frac{\hat{r}}{2}\right) \quad \text{(VI-1)}$$

where S is the initial equity value and $\hat{r}$ **is the dealer's six-month borrowing rate**. With this hedge argument, we had no need to invoke a risk-free interest rate. The role of the dealer's borrowing rate $\hat{r}$ is clear and intuitive.

As an alternative, we may quote the dealer's six-month borrowing rate as if the interest is compounded continuously. Let's now call this continuously compounded borrowing rate $\hat{r}$. In terms of this continuously compounded rate $\hat{r}$, the dealer's payment at expiration is $S\,e^{\hat{r}/2}$. Thus, the solution for the MSFT equity forward price F becomes

$$F \;=\; S\,e^{\hat{r}/2} \quad \text{(VI-2)} \qquad .$$

This form (VI-2) is far less useful to traders than (VI-1). The market does not use continuously compounding interest rates. The continuous compounding is merely a mathematical simplification for many theoretical analyses of derivative pricing.

Example: Equity Forward with Black-Scholes Derivation

We show (VI-2) nonetheless because we can also derive the equity forward price with Black-Scholes analysis. The Black-Scholes partial differential equation and expiration condition for the equity forward are

$$V_t \;=\; \tilde{r}V - x\tilde{r}V_x - \frac{1}{2}\sigma^2 V_{xx}\;,\;\; V(x,T) \;=\; x - F \quad \text{(VI-3)}$$

where x and t are the equity value and time, T is the expiration of the forward contract (six months or half a year), $V(x,t)$ is the value of the forward contract for the investor, $\tilde{r}$ is the continuously compounded "Black-Scholes interest rate" (which we shall discuss) and σ is the equity volatility. The solution of (VI-3) is

$$V(x,t) \;=\; x - F\,e^{-\tilde{r}(T-t)} \quad \text{(VI-4)} \qquad .$$

To get the forward price F at contract inception, we set $t=0$, $x=S$, $T=1/2$, and $V(S,0)=0$. The resulting Black-Scholes solution for the forward price F is

$$F \;=\; S\,e^{\tilde{r}/2} \quad \text{(VI-5)} \qquad .$$

For continuously compounded interest rates, the hedge solution (VI-2) is identical to the Black-Scholes solution (VI-5) if one assumes that the interest rates $\hat{r}$ and $\tilde{r}$ are equal. The former is the dealer's borrowing cost while quantitative analysts call the latter the "risk-free rate". For the Black-Scholes result to be correct, the "risk-free rate" must, in fact, be the dealer's borrowing rate $\hat{r}$ since (VI-2) describes the forward equity price F at which the market will execute the equity forward.

Example: Equity Forward with Illiquid Equity

We modify the previous equity forward example in two ways. Imagine first that the equity issuer is a Malaysian company and that only Malaysian investors may buy or sell the stock. Thus, the non-Malaysian dealer cannot buy the underlying stock. Second, this equity forward will be cash settled at the six-month expiry. The cash settlement implies that the investor will receive a payment for the market (bid) value of the stock rather than the stock itself in exchange for the investor's payment of F to the dealer.

The dealer cannot use "hedge construction" to determine the appropriate forward price F in this situation since the dealer cannot buy the underlying stock. Thus, the dealer can only execute the trade if it is willing to bear the risk of a short position in the Malaysian stock. Without the ability to hedge, there is no methodical procedure to derive F for the trade. The dealer and investor may agree on a forward price, but it's highly likely the dealer will require a forward price greater than the future expected value of the stock as compensation for the dealer's risk of loss. (The dealer will assess equity capital against the risk of loss in this trade. The dealer's expected profit must provide a healthy return on the equity capital.)

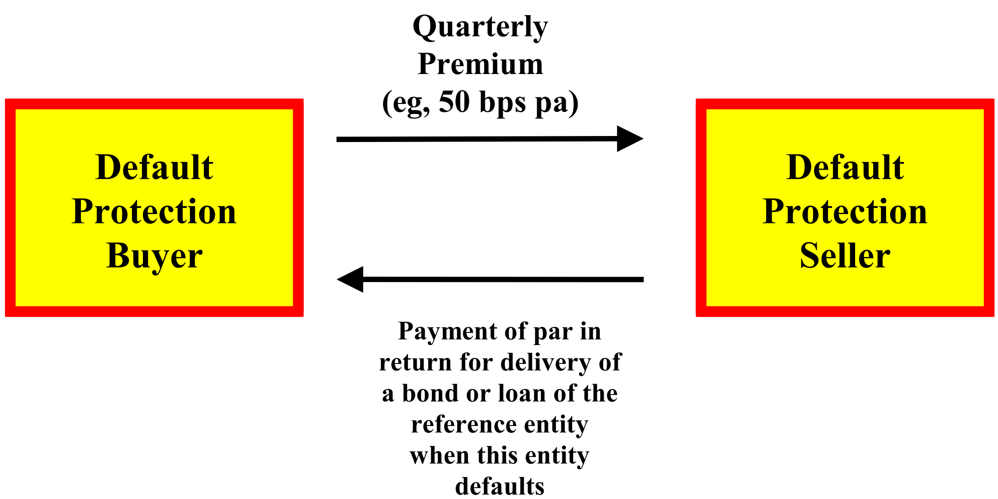
The inability to purchase the Malaysian stock does not, however, obviously diminish the concept of risk neutrality. The stock has an observable price S in the market and one can utter the phrase "a risk-neutral investor is indifferent to the risk of the equity" and would therefore value this equity forward as if the equity drift were the "risk-free" rate $\tilde{r}$. Hence, the risk neutrality argument would continue to find the forward price F to be $F = S\, e^{\tilde{r}/2}$. This result is "wrong" in that no rational dealer would accept this value for F even with the understanding that the interest rate $\tilde{r}$ is the dealer's borrowing cost.

From this discussion and from the broader construction of the Black-Scholes model for derivative pricing, it is clear that the concept of risk neutrality is wrong in the absence of a hedge.

Example: Credit Default Swaps

An excellent case study in the delusion of risk neutrality as a derivative pricing methodology arises from the study of the credit default swap. This instrument is a financial contract between two parties (the "buyer" and "seller"). The buyer pays a quarterly premium to the seller over the life of the trade. The seller pays nothing in return unless a debt obligation of a specified "reference entity" defaults. If default occurs, then the trade terminates and the seller must pay to the buyer the notional amount of the contract while the buyer delivers to the seller a debt obligation of the reference entity with face value equal to the notional amount. Since this reference entity has defaulted on one of its debt obligations, it is highly likely that the seller will suffer a substantial loss upon this default and early termination. That is, the bond or loan that the buyer delivers to the seller will likely be worth much less than the par value that the seller pays to the buyer.

This credit default swap is essentially an insurance contract. The buyer pays the premium and collects an economic loss amount if and when the reference entity defaults.



Though this credit derivative instrument has been widely traded for several years, the models for pricing the credit default swap vary widely. We discuss one of these models here.

Risk neutrality is the basic principle of the dominant academic approach to credit derivative pricing.[5-8] Proponents of the risk neutrality approach construct impressive mathematical edifices, but we can outline here in a simple and direct manner the important aspects.

Risk Neutrality Argument for Credit Default Swap Pricing

The following discussion gives the standard risk-neutral treatment of credit default swap pricing. We wish to find the "fair premium" for a one-year credit default swap in which IBM is the reference entity. The one-year US Treasury yield is r. The yield of a one-year (remaining maturity) IBM bond trading in the secondary market is $r+s$ where we call s the "spread to Treasuries" of the IBM bond. For simplicity, we think in terms of a single payment period of length 1 year. Another simplifying assumption is that we recognize default only at the end of the period. An investor may purchase the risk-free Treasury security and earn $1+r$ or purchase the risky IBM bond and earn $1+r+s$ if there is no default or earn $R(1+r+s)$ if IBM does default. This parameter R is the "recovery rate" of the bond and has an unknown value between 0 and 1. Models of this sort generally require the user to specify the recovery rate R even though one has little confidence in the prediction of this parameter.

Since a "risk-neutral investor" desires the same expected return for each investment regardless of risk, we can use the Treasury spread s of the IBM bond to infer the "risk-neutral default probability" p of the bond. That is, we know the expected return of the IBM bond is the IBM default probability multiplied by the return given default plus the product of the IBM survival probability and the return when there is no default. Since this IBM bond expected return equals the 1-year Treasury's expected return for the "risk-neutral investor", we write

$$1+r \ = \ pR(1+r+s)+(1-p)(1+r+s) \quad \text{(VI-6)} \qquad .$$

Bear in mind that this "risk-neutral default probability" is a wholly fictitious quantity. It is the default probability that would be consistent with the market prices of the Treasury and IBM bonds if investors did not require additional compensation for variability of return. Since investors do require this compensation, the "risk-neutral default probability" is certainly not equal to the "true default probability" of the bond. But, if everything works as expected, this "risk-neutral default probability" will give correct derivative pricing.

We solve equation (VI-6) to find this risk-neutral default probability as

$$p \ = \ \frac{s}{(1-R)(1+r+s)} \quad \text{(VI-7)} \qquad .$$

To get the fair default swap premium with this risk neutral method, we observe that the buyer's expected gain in the contract must balance the premium it pays. The expected gain is the risk-neutral default probability p multiplied by the gain upon default. The gain upon default is $1-R(1+r+s)$. With this expression and (VI-7), the fair premium is*

$$\frac{s}{1+r+s}\left[1-\frac{R(r+s)}{1-R}\right]$$

For typical values of parameters, this fair premium is "almost equal" to the Treasury spread s. We've reached the central result of risk neutrality pricing for

* Economically, the credit default swap should require the seller of protection to pay par plus accrued interest upon a default of the reference entity rather than just par. The buyer's gain upon default would then be $(1-R)(1+r+s)$ and, more importantly, the fair premium would be the spread to Treasuries s.

credit default swaps: the fair premium is (approximately) equal to the risky bond spread to the (risk-free) US Treasury curve.

Problem with the Risk Neutrality Result for Credit Default Swap Pricing

The problem with the risk neutrality result for credit default swaps (*i.e.*, that the fair premium is approximately equal to the risky bond's spread to the Treasury curve) is that it's wrong. More to the point, it disagrees markedly with the market's valuation of default swaps. The market default swap premium corresponds strongly to the spread to the LIBOR curve of the risky bonds of the reference entity. LIBOR and Treasury rates are quite different.

Notice that the preceding risk-neutral pricing argument derived a "risk-neutral default probability", but there was no discussion of a hedged portfolio. In fact, there was no hedge at all. The argument simply assumed that one should imagine that an investor would buy a risky bond and demand an expected return equal to the risk-free rate r.

This omission of the hedge is a critical error. Without the hedge, there is no justification for risk neutrality - which we prefer to call "risk equality". Further, it is the hedge argument itself that shows that the risk-free rate r is irrelevant. Though the conclusion may not be intuitively appealing, the Treasury rate (for USD currency trades) has absolutely no bearing on credit derivative pricing.

Some purveyors of these risk-neutral models attempt to repair their efforts by declaring that traders should substitute LIBOR for the US Treasury curve as the risk-free rate.[6,9] This modification "works" in the sense that the model then gives results consistent with the credit derivative market. But the model is no longer credible since LIBOR is clearly not the risk-free rate.

Another attempt to explain the apparent failure of risk neutrality is a dissection of the meaning of the yield spread between a risky corporate bond and the US Treasury curve. The argument is that a bond investor receives the additional spread over the Treasury yield to compensate him/her both for the bond's credit risk and for the bond's relative illiquidity. Thus, to price a credit derivative, some practitioners argue that one must apportion some of the Treasury spread to credit risk and some to illiqudity. The credit default swap premium should then be approximately equal to the portion of this spread due to credit risk. The analysts in this school of thought are searching for a "risk-free, but illiquid" yield curve benchmark.[9]

We reject this hypothesis of "liquidity deconstruction" for two reasons. First, if the market pays an investor in an IBM bond extra yield for liquidity, then the seller of protection in a credit default swap referencing IBM should receive liquidity compensation as well. Splitting off a liquidity component for one instrument only makes no sense. Second, it is straightforward to construct a hedge model for the credit default swap in which one finds quite naturally and intuitively that the default swap premium should be approximately the risky bond's spread to the LIBOR curve. Since the hedge construction itself demonstrates why the default swap premium tracks the LIBOR spread much more closely than the spread to the risk-free rate, there is no reason to postulate a liquidity effect. The issue of liquidity need never arise.

VII. Summary

There are two primary, competing views of credit default swap pricing. One view is highly mathematical, pedantic, claims to apply risk neutrality, and finds a "spread to Treasuries" result for the default swap premium. The other view is much simpler, relies on the dealer's hedge construction, and gives a "spread to LIBOR" outcome. It is critically important to the credit derivative field that finance analysts sort out which view is correct and why.

We have argued in this article that the "spread to Treasuries" view is in error due to a mistaken application of risk neutrality. Further evidence is the lack of agreement between "spread to Treasuries" and the levels at which the market prices credit default swaps. We devoted the primary portion of this discussion to the general concept of risk neutrality rather than to the more specific issue of credit derivative pricing.

Our analysis of the Black-Scholes and Cox-Ross trailblazing research efforts found that the risk-free rate is irrelevant to virtually all derivative pricing. It is the dealer funding cost - which we often approximate as LIBOR - that is the proper interest rate for derivative pricing. We also interpret Harrison-Pliska as supportive of our view that dealer funding cost rather than risk-free rate should play the key role in mathematical finance derivations.

Further, what the industry calls "risk neutrality" exists only if a dealer can effectively hedge the derivative product in question. If a hedge does not exist, risk neutrality is absent. Black-Scholes and Cox-Ross both made very explicit the importance of the hedge existence. We suggest, then, that all derivative pricing derivations must clearly explain the hedge and how it enters the pricing model.

Finally, we suggested that "risk equality" is a better name for the mathematical trick that the industry now calls "risk neutrality". The latter name is misleading. There is no "world" in which investors demand no compensation for risk. There is only the "trick" that the expected derivative instrument return will equal the dealer funding cost when one forces the expected return of the underlying market variable to be equal to this same funding cost.

To support these claims we reviewed the Black-Scholes and Cox-Ross studies. We also demonstrated with simple examples (the equity forward and the credit default swap) the alternative approaches of risk neutrality and hedge construction. The positions in this document do not directly contradict any aspect of Black-Scholes, Cox-Ross, or Harrison-Pliska. Rather, none of these articles defined the "riskless" rate unambiguously. The Cox-Ross statement that the "riskless" rate is "a constant rate at which individuals can borrow and lend freely" is consistent with our belief.

The great legacies of Black-Scholes and Cox-Ross are the concept of the hedged portfolio and some mathematical techniques to exploit that insight. Much subsequent work in finance has emphasized the mathematics at the expense of the humbler role of the hedged portfolio. In cases such as credit default swap pricing, the omission of the hedge gives wrong answers.

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