Simplifying Expression for the Classical Bond Price-Yield Relationship

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The "classical bond math" approach to valuing a fixed-income debt obligation is ubiquitous in the financial world. In this framework, one may convert a bond price, p, to a bond yield, y, and vice-versa.

Both price *and* yield are critical parameters for the bond trader and risk manager in different contexts. The price provides the actual purchase or sale proceeds. Ultimate return in a trading strategy depends on prices (and times) at which the trader buys and sells.

Bond yield, on the other hand, is paramount in comparing one bond to another. Specifically, two bonds of the same obligor, payment priority, collateral security and remaining maturity may have vastly different prices, but the yields must be nearly equal. Even for two bonds that differ in essentially every respect, it is the yield measure, rather than the price measure, that provides the more intuitive sense of relative value.

Hence, the analyst must know and consider both the price and yield of a bond. Our term, "classical bond math" (CBM), denotes the well-known method for performing the conversion from price to yield and yield to price. We describe in this article a new result that aids the interpretation of a general case.

Bond Price with Conventional Discount Yield

Let us consider first a standard case in which a bond with principal amount P_0 pays interest to the bearer equal to $c\Delta t$ for every time period Δt from the present to maturity $T = N\Delta t$, where N is the number of interest ("coupon") payments. The period Δt is measured in years. Thus, semi-annual bond payments imply $\Delta t = 1/2$. The coupon c is a per-annum fraction of par (principal). Hence, a c value of 0.05 means the investor receives interest of 5% of the principal amount per annum.

In this communication, we assume the first coupon period and all subsequent periods are of the same period length Δt. The generalization to an arbitrary first-period length is straightforward. But we wish to avoid the discussions of "stub periods" or of "accrued interest" that such generalization would require, since the general result does not impact our final result of a simplifying CBM price-yield relationship.

With these prescriptions, equation (1) gives the price ("value") of the bond as a function of yield y, as follows:

$$\text{price } p(y) = P_0\, c\, \Delta t \sum_{i=1}^{N} \alpha^i + P_0 \alpha^N \quad \text{with} \quad \alpha \equiv \frac{1}{1 + y\Delta t} \; . \quad (1)$$

Note that yield y enters as the key parameter for the time-dependent discount factor α. This encapsulation of discount factor value at all forward times, within a single value y, is the hallmark of CBM. Analysts understand it to be a great simplification of reality, but the convenience of the representation is overwhelming. One may readily evaluate the summation of equation (1) to find:

$$p(y) = P_0 \frac{\alpha c \Delta t}{1-\alpha}\left(1 - \alpha^N\right) + P_0 \alpha^N \quad (2a)$$

$$\text{and } p(y) = P_0 + P_0\left(\frac{c-y}{y}\right)\left(1 - \alpha^N\right) \; . \quad (2b)$$

Equation (2b) expresses the "benchmark result" that a bond trades at par (i.e., the bond value equals the outstanding principal amount) when the coupon rate c equals the bond yield y. This $c = y$ outcome is independent of coupon period Δt and maturity T.

Bond Price for Arbitrary Coupon and Principal Amortization

Our goal here is to find an analogous result for the more general bond with amortizing principal and unequal coupon rates. The previous example was a "bullet bond" that returned all principal to the investor at the final payment date at time $T = N\Delta t$. Some bonds – e.g., mortgage bonds, sinking fund bonds and pass-through bonds – repay principal at each payment date rather than simply at the final date.

To consider this more general bond structure, we now write P_i as the outstanding principal prevailing just *after* the payment at time $i\Delta t$. With P_0 still the initial outstanding principal, it follows that the principal payment at time $i\Delta t$ is $P_{i-1} - P_i$.

Since the bond matures at time $T = N\Delta t$, then $P_N = 0$ (since P_N is the outstanding principal after the payment at time T). We permit the coupon rate to be variable by writing it as c_i rather than just c. By the interest-in-arrears convention of bond payments, the interest payment at time $i\Delta t$ is $P_{i-1}\, c_{i-1} \Delta t$.

In a manner identical to equation (1), we write the bond price as the sum of discounted interest and principal payments, and find the following:

$$p(y) = \sum_{i=1}^{N} \left[P_{i-1} c_{i-1} \Delta t + (P_{i-1} - P_i)\right] \alpha^i \quad (3)$$

Manipulating one of these terms,

$$\sum_{i=1}^{N} \alpha^i (P_{i-1} - P_i) = \alpha \sum_{i=1}^{N} \alpha^{i-1} P_{i-1} - \sum_{i=1}^{N} \alpha^i P_i$$

$$= \alpha \sum_{i=0}^{N-1} \alpha^i P_i - \sum_{i=1}^{N} \alpha^i P_i = \alpha P_0 - (1-\alpha) \sum_{i=1}^{N-1} \alpha^i P_i \; , \quad (4)$$

we continue to the final result:

$$p(y) = P_0 + \alpha \Delta t \sum_{i=0}^{N-1} \alpha^i P_i (c_i - y) \; . \quad (5)$$

The interest and importance of equation (5) is that we see the familiar emphasis on the difference between coupon rate and yield. If we specify a constant coupon rate $c_i = c$ while maintaining arbitrary principal amortization, then the "par value condition" $c = y$ still holds. When we maintain the variability of coupon rate c_i, equation (5) shows that the par condition holds when yield y is a suitable weighted average of the c_i, as follows:

$$y = \sum_{i=0}^{N-1} \alpha^i P_i c_i \Bigg/ \sum_{i=0}^{N-1} \alpha^i P_i$$

This encapsulation of discount factor value at all forward times, within a single value y, is the hallmark of classical bond math. Though a great simplification of reality, the convenience of the representation is overwhelming.

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