

Yield Curve Calculations

Background: Everything is “discount factors”

Yield curve calculations include valuation of forward rate agreements (FRAs), swaps, interest rate options, and forward rates. The most important component of all these calculations is the determination of “zero coupon discount factors” (or, just “discount factors”). This note focuses on the problem of computing the forward rate between any two future dates. This ordeal effectively ends when we get the discount factors for the two dates since the remaining labor for the forward rate is straightforward.

Discount factors are synonymous with “present value”

Each forward date has an associated discount factor that represents the value today of a hypothetical payment that one would receive on the forward date expressed as a fraction of the hypothetical payment.* For example, if we expect to receive \$1000 in six months, then its present value might be \$989. Thus, the discount factor for six months would be 0.989. The question then becomes how to derive the \$989 of this example.

Before we begin, let’s remark that discount factors are all about interest rates. Unless stated otherwise (very unlikely in this article!), we deal here with the “LIBOR/swap curve”. The meaning of this term will emerge in the following sections.

Begin with the simplest calculations for discount factors

The discount factor for a forward point in time is the value today (also known as “present value”) of a payment to be received at the forward time expressed as a fraction. By definition, then, the discount factor for “today” is 1.0 since a payment today has present value of precisely the payment value. In the example of the preceding section, we made up the number \$989 as the present value of a payment of \$1000 to be received in six months. The connection between these two amounts is investment with interest. If we have \$989 today, we can invest this amount for six months. If in six months’ time our original principal and interest totals \$1000, then we are indifferent to having \$989 now or \$1000 in six months.

Let L_6 be the interest rate (quoted annually) at which we can invest for six months. Then the interest earned will be $\$989 * L_6 * \frac{1}{2}$ where we multiply

* Please don’t take the word “today” too literally. Due to market convention, “today” in some contexts means two business days forward from “today”. This is “spot settlement” and we choose not to elevate the importance of this distinction here.

by one-half to denote six months (half a year). Adding the principal and interest and setting the sum to the \$1000 value at six months shows

$$\$989 + \$989 * L_6 * \frac{1}{2} = \$1000 \quad .$$

Re-arranging terms and expressing as a fraction of ultimate payment gives

$$0.989 = \frac{1}{1 + L_6 / 2} \quad .$$

The equation above is somewhat backward since we have a constant value on the left-hand side and what appears to be a variable (L_6) on the right-hand side. In real life, though, we will know L_6 and will need to determine the discount factor which, in this example, we stipulated to be 0.989.

We've over-simplified. We need to take into account the "daycount convention" of the interest rates we use. The L_6 - which we will shortly describe as "six-month LIBOR" - has the daycount of "actual/360" which means we multiply by the number of days in the period and divide by 360 (as if each year has 360 days). In a six-month time period, the actual daycount is more likely to be 182 than 180. Our convention will be to use Δt - often with a subscript - to indicate the time period *in years* between two dates measured by the governing daycount convention. To denote both aspects that the discount factor is the value to be calculated and that that we need to determine the precise Δt as per the daycount convention, we re-write the previous equation as

$$Discount\ factor = \frac{1}{1 + L_6 \Delta t} \quad .$$

Let's review the available market information

To compute discount factors, we begin with market interest rate information. The market rates of interest to us are LIBOR (the "London inter-bank offered rate") for maturities of twelve months and less and "swap rates" for maturities of two years to thirty years. (See *Bloomberg USSWAP*.) The most prominent LIBOR maturities are 1 month, 3 months, 6 months, and 12 months, though there are also settings for every other month, overnight, 1 week and 2 weeks. (See *Bloomberg BBAM*.)

We ignore *Eurodollar* futures contracts primarily for convenience. They complicate the analysis significantly. In principle, the swap data we use will give us almost identical results to what we'd expect with futures.*

All LIBOR settings imply simple interest with actual/360 daycount. Swaps are not so accommodating. In a swap, one side pays a fixed rate with 30/360 daycount semi-annually. The other side pays 3-month LIBOR with

* The difference in final results is "convexity" and is important to professional swap dealers.

actual/360 daycount *quarterly*. A quoted swap rate of, say, 4% for a 5-year maturity means that a 5-year swap in which the fixed-rate payer pays 4% semi-annually and the floating-rate payer pays 3-month LIBOR quarterly will have a zero value (*i.e.*, the swap is at-market).

Re-visit the simplest calculations for discount factors

The earlier section that discussed “simplest calculations for discount factors” treated only the case with a 6-month maturity. The calculation is nearly identical for any maturity that matches that of an available LIBOR maturity. For example, the discount factor at 3 months is just

$$\text{Discount factor} = \frac{1}{1 + L_3 \Delta t}$$

in which the Δt is the actual number of days to the forward date (likely 90, 91, or 92) divided by 360. This method “works” for any LIBOR maturity. But such maturities do not extend beyond 12 months and it is this point at which the discount factor calculation becomes more challenging.

Linear interpolation for yields

Before tackling the challenging problem of discount factors for maturities beyond 12 months, let’s imagine we want the discount factor for 6.5 months. There is no 6.5-month LIBOR, so we apply the expression

$$\text{Discount factor} = \frac{1}{1 + L \Delta t} .$$

Here we designate L as the linear interpolation of the 6-month and 7-month LIBOR settings.

Generally speaking, linear interpolation works well for yields but is not appropriate for discount factors. Based on the mathematical nature of how the discount factor depends on yield, linear interpolation of yield suggests logarithmic interpolation for discount factors. Hence, we view the latter as the proper choice for discount factor interpolation when necessary.

It’s not uncommon to hear others promote (cubic or higher order) splines for yield interpolation. Our view is that splines do not provide greater accuracy and, in fact, introduce numerical risk. Splines may be more exotic, but that doesn’t make them better.

Bootstrapping for discount factors beyond 12 months

Beyond 12 months we must rely on swap data. Swaps behave much differently than simple borrowing and lending at LIBOR. As we noted previously, the fixed-rate payer pays the swap (fixed) rate semi-annually and receives 3-month LIBOR quarterly (from the floating-rate payer). So the swap rate is, roughly speaking, an average of 3-month LIBOR from “now” until the swap contract maturity.

To compute discount factors we adopt a recursive procedure known as “bootstrapping”. Our goal is to compute discount factors at six-month intervals. We know the discount factors at six and twelve months from the simple-interest LIBOR calculations. For the discount factor at eighteen months, we use both the (fixed) swap rate R_3 and the two prior discount factors z_1 and z_2 to compute the 18-month discount factor z_3 . Remember that the discount factor for “today” is $z_0 = 1$. Let f_j be the forward LIBOR rate for the interval t_{j-1} to t_j . As we’ll note later, “forward” LIBOR is the expected future value of LIBOR given today’s market. Then, we write the market value of the swap as the present value of expected future cashflows. Since a swap “at-market” has zero value, we get

$$0 = \sum_{j=1}^i \left(-R_i z_j \Delta \hat{t}_j + f_j z_j \Delta t_j \right) \quad (1) .$$

(To get z_3 , the discount factor at 18 months, we set $i = 3$ in this equation.) We define $\Delta t_j = t_j - t_{j-1}$ with the actual/360 daycount convention and $\Delta \hat{t}_j = t_j - t_{j-1}$ with the 30/360 daycount convention.

Before proceeding, we must make a remarkable observation. The second term in the summation of equation (1) simplifies drastically to

$$\sum_{j=1}^i f_j z_j \Delta t_j = z_0 - z_i \quad (2) .$$

Let’s call this the “algebraic wonder of finance” and it explains why LIBOR floaters reset to par with each (quarterly) rate setting. Of course, the math doesn’t really drive the finance. Rather, it’s the underlying “physics” of LIBOR floaters that manifests itself in the wonder of equation (2). The curious reader may verify equation (2) given the forward rate f_j that emerges from an arbitrage argument:

$$f_j = \frac{z_{j-1} - z_j}{z_j \Delta t_j} \quad (3) .$$

Combining equations (1) and (2) and realizing that $z_0 = 1$ and $\Delta \hat{t}_j = 1/2$, we can solve for z_i as

$$z_i = \frac{2 - R_i \sum_{j=1}^{i-1} z_j}{2 + R_i} \quad (4) .$$

This is bootstrapping! Equation (4) gives us z_3 , for example, when we know z_1 , z_2 , and R_3 . Once we do have z_3 , we’d then compute z_4 and so on.

There are three loose ends here. First, we need to have swap rates R_i at all 6-month maturity points beginning at 18 months. The market does not quote this many swap rates. Where necessary, then, we interpolate linearly to fill in missing swap rates. Second, we get discount factors only at these 6-month maturity points. When we need a discount factor at any other time point, we interpolate logarithmically as we discussed earlier. Finally, our equation (1) seems to have forgotten that the floating rate payments in a swap are quarterly. We simplified the problem by making the floating rate payments semi-annual. When we treat the payment frequency correctly, we do get the same results in the ensuing equations. I removed a layer of complexity by the approximate method of equation (1).

Arbitrage argument for the forward rate

Equation (3) relates discount factors to forward rates and is central to “the new bond math”.^{*} How does one derive equation (3)? Let’s think of the forward rate f between two future dates as the expected future rate linking the dates. For example, if I agree now to lend you \$1 six months from today and you agree to re-pay me \$1 plus interest nine months from today, then I am making a 3-month loan 6 months forward. If you and I knew today what 3-month LIBOR would be in 6 months, then we would use this future LIBOR value as the interest rate for the loan. But we don’t know what 3-month LIBOR will be! Thus, we stipulate the forward rate as the interest rate for the forward loan.

One would think that you and I should negotiate the forward rate based on what we believe 3-month LIBOR will be in 6 months. But that’s not right! Instead, the market tells us forward LIBOR by a clever argument. Instead of making the forward loan of the prior paragraph to you, I will go to a bank and lend for 9 months today at 9-month LIBOR and borrow for 6 months today at 6-month LIBOR. Instead of a forward loan, I have two simple, old-fashioned loans (with one “long” and one “short”). Both loans are in the amount of $\$1/(1+L_6t_6)$.[#] Since the loans are of the same amount, I have no net payment today. My loan cancels my borrowing. But I must re-pay precisely \$1 in 6 months. In 9 months I will receive $(1+L_9t_9)/(1+L_6t_6)$ as repayment for the 9-month loan.

With my two simple loans I have created the (synthetic) forward loan. Now I can compute the effective interest rate of this forward loan with

$$(1+L_9t_9)/(1+L_6t_6) = 1+f(t_9-t_6) \quad (5) .$$

^{*} In contrast, “the old bond math” is comprised of “yield-to-maturity” and expressions of the form $(1+y/2)^{2n}$ to compute forward values.

[#] I hope the notation isn’t confusing. The subscripts “6” and “9” refer to six and nine months, respectively. The t is, once again, time as fraction of a year in the actual/360 daycount convention.

Solving equation (5) for f with $\Delta t = (t_2 - t_1)$ and relating discount factors to the LIBOR values gives equation (3). This argument applies for any two dates rather than just those less than 12 months for which we can use LIBOR values as we've done here. For the dates t_1 and t_2 , we would create the synthetic forward loan by borrowing an amount z_1 for time t_1 and lending this same amount z_1 for time t_2 . At time t_1 we'd then pay \$1 while at time t_2 we'd receive z_1/z_2 . Equation (5) would then become

$$z_1/z_2 = 1 + f(t_2 - t_1) \quad (6) .$$

Meaning of the forward rate

It's not uncommon to hear market players disparage forward rates with a comment such as "forward rates are not good predictors of future rates". When you hear this statement, brace yourself! You're about to be assailed with an heroic market story in which the speaker-protagonist shrewdly predicted some past market move. If the speaker is your boss, then nod appreciatively. If not, smirk derisively.

The forward rate is the market's expectation of the future rate. It's not a prediction in the sense that any of us should believe that LIBOR will be this value. Rather, the forward rate is the consensus average value of potential future outcomes. For illustration, if the forward rate for a particular period is 4% and you believe rates will, in reality, be higher (lower) at this future time, you can place a bet (with a *Eurodollar* futures contract). If LIBOR in the future is higher (lower) than 4%, you win. Conversely, if LIBOR in the future is lower (higher) than 4%, you lose. If LIBOR actually does come in at 4%, you break even.

The forward rate, then, is not just a calculation and it's not the output of some econometric model. It's real! The forward rate for a defined future period is just as much a market variable as 3-month LIBOR or the IBM stock price. The current value of any market variable is the market consensus. It's neither right nor wrong, it's the market!